Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
Prof. Dr. J. Struckmeier
Dr. H. P. Kiani

Differential Equations II for Engineering Students

Work sheet 6

Exercise 1:

From the lecture we know d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\alpha)\,d\alpha$$

for solving the initial value problem for the (homogeneous) wave equation

$$\hat{u}_{tt} - c^2\hat{u}_{xx} = 0,\ \hat{u}(x,0) = g(x),\ \hat{u}_t(x,0) = h(x),\ x \in \mathbb{R},\ c > 0.$$

The function

$$\tilde{u}(x,t) = \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} f(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau \tag{1}$$

solves the following initial value problem

$$\tilde{u}_{tt} - c^2\tilde{u}_{xx} = f(x,t) \qquad \tilde{u}(x,0) = \tilde{u}_t(x,0) = 0\,. \tag{2}$$

(Proof: Leibniz formula for the derivation of parameter-dependent integrals)

Now we consider the initial value problem

$$\begin{aligned} &u_{tt} - 4u_{xx} = 6x\sin t, \qquad x\in\mathbb{R},\ t>0\\ &u(x,0) = x,\ x\in\mathbb{R}, \qquad u_t(x,0) = \sin(x),\ x\in\mathbb{R} \end{aligned} \tag{3}$$

a) Compute a solution $\hat{u}$ to the IVP

$$\begin{aligned} &\hat{u}_{tt} - 4\hat{u}_{xx} = 0, && x\in\mathbb{R},\ t>0\\ &\hat{u}(x,0) = x,\ x\in\mathbb{R}, && \hat{u}_t(x,0) = \sin(x),\ x\in\mathbb{R}. \end{aligned}$$

b) Compute a solution $\tilde{u}$ to the IVP

$$\begin{aligned} &\tilde{u}_{tt} - 4\tilde{u}_{xx} = 6x\sin t, && x\in\mathbb{R},\ t>0\\ &\tilde{u}(x,0) = 0,\ x\in\mathbb{R}, && \tilde{u}_t(x,0) = 0,\ x\in\mathbb{R} \end{aligned}$$

c) By inserting u into the differential equation and checking the initial values, show that $u = \tilde{u} + \hat{u}$ solves the initial value problem (3).

Solution:

a) Solution of the homogeneous differential equation with the non-homogeneous initial values following d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}(x + 2t + x - 2t) + \frac{1}{4}\int_{x-2t}^{x+2t} \sin(\eta)d\eta = x + \frac{1}{2}\sin(x)\sin(2t)$$

b) Solution of the non-homogeneous differential equation with homogeneous initial values

$$\begin{aligned}\tilde{u}(x,t) &= \frac{1}{4}\int_0^t \int_{x+2(\tau-t)}^{x-2(\tau-t)} 6\omega\,\sin(\tau)\,\mathrm{d}\omega\mathrm{d}\tau = \frac{3}{4}\int_0^t \sin(\tau)\,\left[\omega^2\right]_{x+2(\tau-t)}^{x-2(\tau-t)}\mathrm{d}\tau \\ &= \frac{3}{4}\int_0^t \sin(\tau)\,(-8x(\tau-t)\mathrm{d}\tau = 6x\int_0^t t\sin(\tau) - \tau\sin(\tau)\mathrm{d}\tau \\ &= 6xt(1-\cos(t)) + 6x\left[\tau\cos(\tau)\right]_0^t - 6x\int_0^t \cos(\tau)\mathrm{d}\tau = 6x(t - \sin t)\,.\end{aligned}$$

c) The solution of the original problem is the sum of the two partial solutions:

$$u(x,t) = 6x(t - \sin t) + x + \frac{1}{2}\sin(x)\sin(2t)$$

Test:

$u(x,0) = 6x(0 - \sin(0)) + x + \frac{1}{2}\sin(x)\sin(0) = x,$

$u_t(x,t) = 6x(1-\cos(t)) + 0 + \frac{1}{2}\sin(x)2\cos(2t) = 6x(1-\cos(t)) + \sin(x)\cos(2t)$

$u_t(x,0) = 6x(1-\cos(0)) + \sin(x)\cos(0) = \sin(x)$

$u_{xx} = 0 + 0 - \frac{1}{2}\sin(x)\sin(2t)$ (since the first two summands of u are linear in x, one only has to derive the sine term twice).

$u_{tt} = 6x\sin t - 2)\,\sin(x)\sin(2t)$

$u_{tt} - 4u_{xx} = 6x\sin t - 2\,\sin(x)\sin(2t) - 4\cdot(-\frac{1}{2}\,\sin(x)\sin(2t)) = 6x\sin t\,.$

Exercise 2: (Vibrating String)

Solve the initial boundary value problem

$$\begin{aligned} u_{tt} &= c^2 u_{xx} && \text{for} \quad 0 < x < 1,\ t > 0, \\ u(0,t) &= u(1,t) \ = \ 0 && \text{for} \quad t > 0, \\ u(x,0) &= 0 && \text{for} \quad 0 < x < 1, \\ u_t(x,0) &= \begin{cases} 1, & \frac{1}{20} \le x \le \frac{1}{10}, \\ 0 & \text{else.} \end{cases} \end{aligned}$$

Hint: Lecture page 150.

You will receive a series as a solution. Plot the partial sums of the first 20 non-vanishing summands of this series for $c = 2$, $x \in [0,1], t \in [0, 0.4]$ and $t \in [0,2]$.

Solution:

From the lecture (page 150) we know

$$\boxed{u(x,t) = \sum_{k=1}^{n} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x), \qquad \omega = \frac{\pi}{L} = \pi.}$$

Also the initial conditions have to be fulfilled. The first one for $n \to \infty$ yields

$$u(x,0) = \sum_{k=1}^{\infty} \left(A_k \cos(0) + B_k \sin(0)\right) \cdot \sin(k\omega x) = \sum_{k=1}^{\infty} A_k \sin(k\pi x) = 0 \qquad x \in [0,1].$$

So we obtain $A_k = 0, \forall k \in \mathbb{N}$.

The second initial condition is:

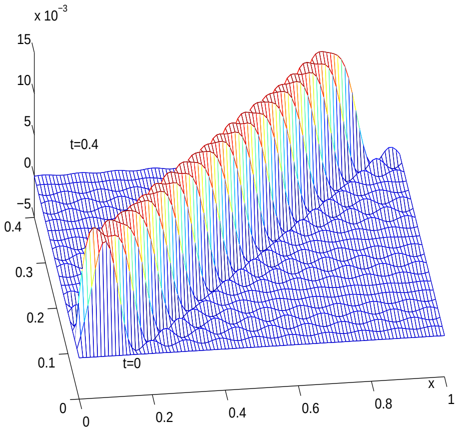
$$u_t(x,0) = \sum_{k=1}^{\infty} ck\pi B_k \cdot \sin(k\pi x) = v_0(x) = \begin{cases} 1, & \frac{1}{20} \le x \le \frac{1}{10} \\ 0 & \text{else} \end{cases} \qquad x \in [0,1].$$

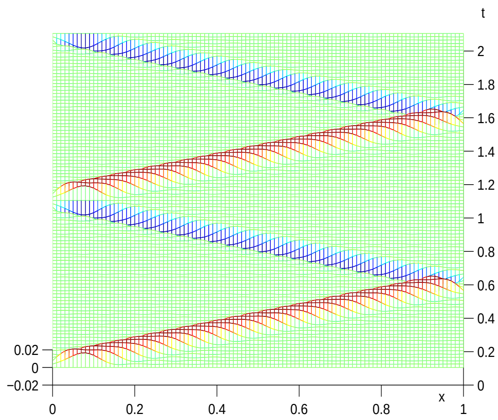
We compute the Fourier coefficients of the odd, 2-periodic continuation of v_0:

$$b_k = 2 \int_0^1 v_0(x) \sin(k\pi x) dx = 2 \int_{\frac{1}{20}}^{\frac{1}{10}} \sin(k\pi x) dx = \frac{2}{k\pi} \left[\cos\left(\frac{k\pi}{20}\right) - \cos\left(\frac{k\pi}{10}\right)\right]$$

With $B_k = \dfrac{b_k}{ck\pi}$ we have

$$\boxed{u(x,t) = \frac{2}{c\pi^2} \sum_{k=1}^{n} \frac{1}{k^2} \left[\cos\left(\frac{k\pi}{20}\right) - \cos\left(\frac{k\pi}{10}\right)\right] \sin(ck\pi t) \cdot \sin(k\pi x)}$$


x 10^-3
15
10
5
0
−5
t=0.4
t=0
0.4
0.3
0.2
0.1
0
0
0.2
0.4
0.6
0.8
1
x


t
2
1.8
1.6
1.4
1.2
1
0.8
0.6
0.4
0.2
0
0.02
0
−0.02
0
0.2
0.4
0.6
0.8
1
x