Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Homework sheet 6

Exercise 1:

a) Solve the initial value problem

$$\begin{aligned} u_{tt} &= u_{xx}, && \text{on } \mathbb{R}^2\,, \\ u(x,0) &= 2\sin(4\pi x) && x \in \mathbb{R}\,, \\ u_t(x,0) &= \cos(\pi x) && x \in \mathbb{R}\,. \end{aligned}$$

b) Consider the problem

$$\begin{aligned} u_{tt} &= 9u_{xx}, \quad \text{for } x \in \mathbb{R},\ t > 0\,, \\ u(x,0) &= u_0(x) = \begin{cases} 2 & -1 \le x \le 1\,, \\ 0 & \text{otherwise,} \end{cases} \\ u_t(x,0) &= 0. \end{aligned}$$

Sketch the solution obtained by d'Alembert's formula for

$t = 0,\ \frac{1}{6},\ \frac{1}{3},\ \frac{2}{3},\ 1\,.$

Solution:

a) Using d'Alembert we have

$$\begin{aligned} u(x,t) &= \sin\left(4\pi(x+t)\right) + \sin\left(4\pi(x-t)\right) + \frac{1}{2c}\int_{x-t}^{x+t} \cos(\pi\eta)\,d\eta \\ &= 2\sin(4\pi x)\cos(4\pi t) + \left.\frac{\sin(\pi\eta)}{2\pi}\right|_{x-t}^{x+t} \\ &= 2\sin(4\pi x)\cos(4\pi t) + \frac{1}{2\pi}\left(\sin\left(\pi(x+t)\right) - \sin\left(\pi(x-t)\right)\right) \\ &= 2\sin(4\pi x)\cos(4\pi t) + \frac{1}{\pi}\left(\cos\left(\pi x\right)\cdot\sin\left(\pi t\right)\right). \end{aligned}$$

b) d'Alembert's formula yields $u(x,t) = \dfrac{1}{2}\left(u_0(x+3t) + u_0(x-3t)\right)$ with

$$u_0(x+3t) = \begin{cases} 2 & -1-3t \le x \le 1-3t\,, \\ 0 & \text{else} \end{cases} \qquad \text{and}$$

$$u_0(x-3t) = \begin{cases} 2 & -1+3t \le x \le 1+3t\,, \\ 0 & \text{else.} \end{cases}$$

So we obtain for $-1+3t \le 1-3t \iff t \le \frac{1}{3}$

$$u(x,t) = \begin{cases} 0 & x < -1-3t, \\ 1 & -1-3t \le x < -1+3t, \\ 2 & -1+3t \le x \le 1-3t, \\ 1 & 1-3t < x \le 1+3t, \\ 0 & 1+3t < x. \end{cases}$$

For $t = 1/6$:

$x - 3t = x - 0.5 \in [-1,1] \iff x \in [-0.5, 1.5]$ and

$x + 3t = x + 0.5 \in [-1,1] \iff x \in [-1.5, 0.5]\,.$

$$u(x,\frac{1}{6}) = \begin{cases} 0 & x < -1.5, \\ 1 & -1.5 \le x < -0.5, \\ 2 & -0.5 \le x \le 0.5, \\ 1 & 0.5 < x \le 1.5, \\ 0 & 1.5 < x. \end{cases}$$

The solution for $t = \frac{1}{3}$ is calculated analogously.

$$u(x,\frac{1}{3}) = \begin{cases} 0 & x < -2, \\ 1 & -2 \le x < 0, \\ 2 & 0 \le x \le 0, \\ 1 & 0 < x \le 2, \\ 0 & 2 < x. \end{cases}$$

For $t > \frac{1}{3}$ we have $1-3t < -1+3t$ and

$$u(x,t) = \begin{cases} 0 & x < -1-3t, \\ 1 & -1-3t \le x < 1-3t, \\ 0 & 1-3t \le x \le -1+3t, \\ 1 & -1+3t < x \le 1+3t, \\ 0 & 1+3t < x. \end{cases}$$

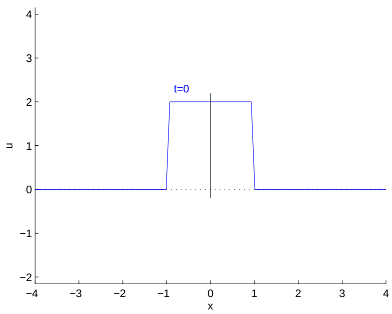
Hence

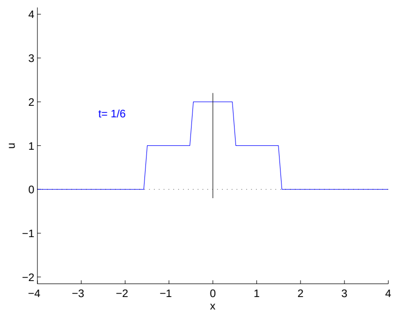
$$u(x, \frac{2}{3}) = \begin{cases} 0 & x < -3, \\ 1 & -3 \le x < -1, \\ 0 & -1 \le x \le 1, \\ 1 & 1 < x \le 3, \\ 0 & 3 < x. \end{cases}$$

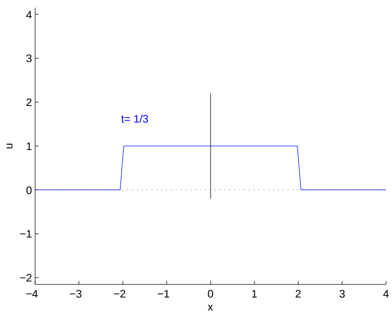
and

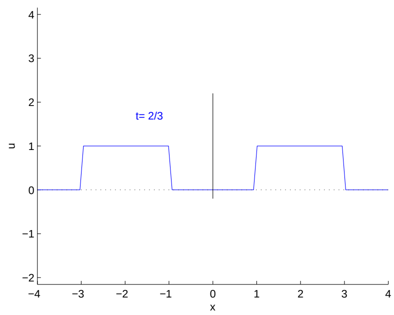
$$u(x, 1) = \begin{cases} 0 & x < -4, \\ 1 & -4 \le x < -2, \\ 0 & -2 \le x \le 2, \\ 1 & 2 < x \le 4, \\ 0 & 4 < x. \end{cases}$$

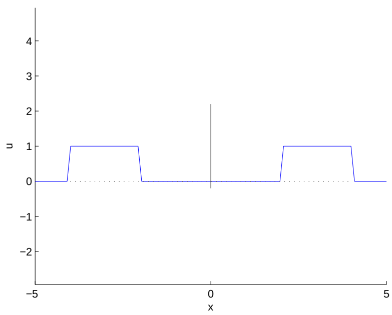
The original (angular) wave clearly breaks up into two waves running in opposite directions.











Exercise 2:

We are looking for an approximation of the solution to the following problem

$$
\begin{aligned}
&u_{tt} = u_{xx} && x \in (0, 2\pi),\ t > 0,\\
&u(x,0) = \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} < x < 2\pi \end{cases} \\
&u_t(x,0) = 0 && x \in (0, 2\pi)\\
&u(0,t) = u(2\pi, t) = 0 && t > 0
\end{aligned}
$$

Sketch the 2π-periodic continuation of the initial data for $x \in [-2\pi, 4\pi]$.

Determine an approximation $\tilde{u}$ to the solution u of the problem using the first three terms of the Fourier series.

Check which boundary and initial conditions are already fulfilled by this approximate solution.

Solution:

General solution:

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x)}$$

With $\omega = \frac{\pi}{L} = \frac{\pi}{2\pi}$ and $c = 1$. Hence

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos(\frac{k}{2}t) + B_k \sin(\frac{k}{2}t)\right) \cdot \sin(\frac{k}{2}x)$$

$$u_t(x,0) = \sum_{k=1}^{\infty} \frac{k}{2} \left(-A_k \sin(0) + B_k \cos(0)\right) \cdot \sin(\frac{k}{2}x) \overset{!}{=} 0 \implies B_k = 0\,,\ \forall k$$

$$u(x,t) = \sum_{k=1}^{\infty} A_K \cos(\frac{k}{2}t) \sin(\frac{k}{2}x)$$

with $u(x,0) = \sum_{k=1}^{\infty} A_K \sin(\frac{k}{2}x) \overset{!}{=} \begin{cases} x & 0 < x < \frac{\pi}{2} \\ \pi - x & \frac{\pi}{2} < x < \frac{3\pi}{2} \\ x - 2\pi & \frac{3\pi}{2} < x < 2\pi \end{cases}$

So we have:

$$
\begin{aligned}
A_k &= \frac{1}{\pi}\int_0^{2\pi} u_0(x)\sin(\frac{k}{2}\,x)\,dx \\
&= \frac{1}{\pi}\left[\int_0^{\pi/2} x\sin(\frac{k}{2}\,x)\,dx \;+\; \int_{\pi/2}^{3\pi/2}(\pi-x)\sin(\frac{k}{2}x)\,dx \;+\; \int_{3\pi/2}^{2\pi}(x-2\pi)\sin(\frac{k}{2}x)\,dx\right] \\
&= \frac{1}{\pi}\left(\left[-x\frac{\cos(\frac{k}{2}x)}{\frac{k}{2}}\right]_0^{\frac{\pi}{2}} \;+\; \int_0^{\pi/2}\frac{2}{k}\cos(\frac{k}{2}x)\;dx\right. \\
&\quad + \left[-(\pi-x)\frac{2}{k}\cos(\frac{k}{2}x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} - \left[(\frac{2}{k})^2\sin(\frac{k}{2}x)\right]_{\frac{\pi}{2}}^{\frac{3\pi}{2}} \\
&\quad \left. - \left[(x-2\pi)\frac{2}{k}\cos(\frac{k}{2}x)\right]_{\frac{3\pi}{2}}^{2\pi} \;+\; \left[(\frac{2}{k})^2\sin(\frac{k}{2}x)\right]_{\frac{3\pi}{2}}^{2\pi}\right) \\
&= \frac{8}{\pi k^2}\left(\sin(\frac{k\pi}{4}) - \sin(\frac{3k\pi}{4})\right).
\end{aligned}
$$

The first three coefficients are the following

$$
\begin{aligned}
A_1 &= \frac{8}{\pi}\left(\sin(\frac{\pi}{4}) - \sin(\frac{3\pi}{4})\right) \;=\; 0, \\
A_2 &= \frac{8}{4\pi}\left(\sin(\frac{\pi}{2}) - \sin(\frac{3\pi}{2})\right) \;=\; \frac{4}{\pi}, \\
A_3 &= \frac{8}{9\pi}\left(\sin(\frac{3\pi}{4}) - \sin(\frac{9\pi}{4})\right) \;=\; 0.
\end{aligned}
$$

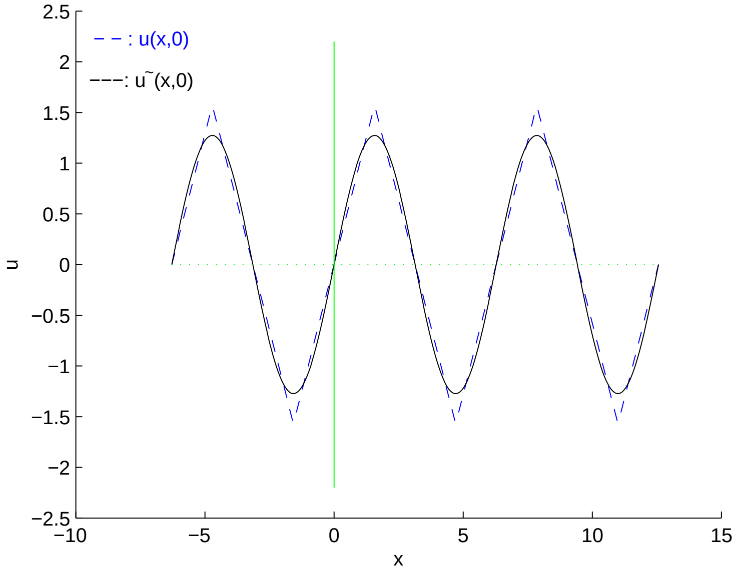
The approximation using the first three terms is given by

$$\tilde{u}(x,t) = u_3(x,t) = \frac{4}{\pi}\cos(t)\sin(x).$$

We have $u_3(x,0) = \dfrac{4}{\pi}\sin(x)$.

The first initial condition for u is hence only approximately fulfilled (see plot). The second initial condition is fulfilled as well as the boundary conditions.

Regarding the boundary conditions, any other outcome would have been a sure indication of a calculation error!


– – : u(x,0)
–––: u˜(x,0)
u
x
2.5
2
1.5
1
0.5
0
−0.5
−1
−1.5
−2
−2.5
−10
−5
0
5
10
15

Exercise 3:

Consider the initial boundary value problem

$$\begin{aligned}
u_{tt} - 4u_{xx} &= e^{-t}\left(1 - \frac{x}{3}\right) && x \in (0,\, 3),\, t > 0,\\
u(x,0) &= 1 + 2\sin(\pi x) && x \in [0,3],\\
u_t(x,0) &= \frac{x}{3} && x \in (0,3), \qquad (1)\\
u(0,t) &= e^{-t} && t \geq 0,\\
u(3,t) &= 1 && t \geq 0.
\end{aligned}$$

Introduce a suitable function v in order to convert the problem into an initial boundary value problem with homogeneous boundary conditions for v.

Give the differential equation and the initial conditions for v.

Solution:

With $v = u - e^{-t} - \frac{x}{3}(1 - e^{-t})$ it holds

$u_t = v_t - e^{-t}(1 - \frac{x}{3}), \quad u_{tt} = v_{tt} + e^{-t}(1 - \frac{x}{3}), \quad v_{xx} = u_{xx}.$

Inserting this into the differential equation we obtain:

$v_{tt} + e^{-t}(1 - \frac{x}{3}) - 4v_{xx} = e^{-t}(1 - \frac{x}{3}) \iff v_{tt} - 4v_{xx} = 0\,.$

Initial and boundary values for v:

$v(x,0) = u(x,0) - e^0 - \frac{x}{3}(1 - e^0) = 1 + 2\sin(\pi x) - 1 = 2\sin(\pi x),$

$v_t(x,0) = u_t(x,0) + e^0 - \frac{x}{3} \cdot e^0 = 1,$

$v(0,t) = v(3,t) = 0\,.$

Discussion: 07.07.-10.07.2025