Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Work sheet 5

Exercise: (See lecture pages 85-90)

We are looking for the solution to the initial boundary value problem (IBVP)

$$\begin{aligned}
u_t - u_{xx} &= e^{-t}\sin(2x) + 1 && x \in (0,\pi)\,, t \in \mathbb{R}^+, \\
u(x,0) &= \frac{1}{2}\sin(2x) && x \in (0,\pi)\,, \\
u(0,t) &= f(t) = t && t \in \mathbb{R}^+ \\
u(\pi,t) &= g(t) = t && t \in \mathbb{R}^+\,.
\end{aligned}$$

a) Homogenize the boundary conditions by using the function

$$v(x,t) = u(x,t) - \left[f(t) + \frac{x}{L}\,(g(t) - f(t)) \right]$$

with $L = \pi$ and replacing the u–expressions with corresponding v–expressions.

b) Solve the following initial boundary value problem analogously to the procedure in the lecture

$$\begin{aligned}
v^*_t - v^*_{xx} &= 0 && x \in (0,\pi)\,, t \in \mathbb{R}^+, \\
v^*(x,0) &= \frac{1}{2}\sin(2x) && x \in (0,\pi)\,, \\
v^*(0,t) &= v^*(\pi,t) = 0 && t \in \mathbb{R}^+\,.
\end{aligned}$$

c) Solve the initial boundary value problem

$$\begin{aligned}
v^{**}_t - v^{**}_{xx} &= e^{-t}\sin(2x) && x \in (0,\pi)\,, t \in \mathbb{R}^+, \\
v^{**}(x,0) &= 0 && x \in (0,\pi)\,, \\
v^{**}(0,t) &= v^{**}(\pi,t) = 0 && t \in \mathbb{R}^+\,.
\end{aligned}$$

using the ansatz

$$v^{**} = \sum_{k=1}^{m} a_k(t)\sin(kx)\,, \qquad a_k(0) = 0$$

d) Give the solution to the initial boundary value problem for u.

Solution:

a) For $\quad v(x,t) = u(x,t) - t - \dfrac{x-0}{\pi-0}(t-t) = u(x,t) - t$

we get $u(x,t) = v(x,t) + t$ and obtain

$u_t = v_t + 1, \quad u_{xx} = v_{xx}\,.$

New PDE:

$v_t + 1 - v_{xx} = e^{-t}\sin(2x) + 1 \iff \boxed{v_t - v_{xx} = e^{-t}\sin(2x).}$

Initial values for v: $v(x,0) = u(x,0) - 0 \overset{!}{=} \dfrac{1}{2}\sin(2x) \qquad x \in (0,\pi)\,.$

Boundary values for v: $v(0,t) = v(\pi,t) = t - t = 0\,.$

b) For the homogeneous differential equation with homogeneous boundary data, $c = 1$ and given initial values

$$v^*(x,0) = \frac{1}{2}\sin(2x), \qquad x \in (0,\pi)\,.$$

Using the formula from page 90 of the lecture

$$v^*(x,t) = \sum_{k=1}^{\infty} a_k\, e^{-k^2 t}\sin(kx)$$

the initial condition reads

$$v^*(x,0) = \sum_{k=1}^{\infty} a_k \sin(kx) = \frac{1}{2}\sin(2x).$$

Comparison of the coefficients gives

$a_2 = \frac{1}{2} \qquad$ and $\qquad a_k = 0,\ \forall k \neq 2\,.$

Hence we obtain

$v^*(x,t) = \dfrac{1}{2}e^{-4t}\sin(2x)\,.$

c) Inhomogeneous differential equation with homogeneous initial and boundary data

$$\begin{aligned} &v^{**}_t - v^{**}_{xx} = e^{-t}\sin(2x) && x \in (0,\pi)\,, t \in \mathbb{R}^+, \\ &v^{**}(x,0) = 0 && x \in (0,\pi)\,, \\ &v^{**}(0,t) = v^{**}(\pi,t) = 0 && t \in \mathbb{R}^+\,. \end{aligned}$$

Ansatz:

$$v^{**} = \sum_{k=1}^{\infty} a_k(t)\sin(kx)\,, \qquad a_k(0) = 0.$$

From the initial condition

$$v^{**}(x,0) = \sum_{k=1}^{\infty} a_k(0)\sin(kx) = 0$$

we conclude $\quad a_k(0) = 0\,.$

Inserting the ansatz into the PDE returns

$$\sum_{k=1}^{\infty} \left[\dot a_k(t) + k^2 a_k(t)\right] \sin(kx) \;=\; e^{-t}\sin(2x).$$

Hence we obtain $a_k(t) \equiv 0$ for $k \neq 2$ and the ODE

$$\dot a_2(t) + 4a_2(t) \;=\; e^{-t}$$

for a_2. The solution to the associated homogeneous equation is

$$a_{2,h}(t) = Ce^{-4t}$$

The ansatz $a_2(t) = C(t)e^{-4t}$ for a solution of the inhomogeneous ODE gives

$$\dot C(t)\,e^{-4t} \;=\; e^{-t} \quad\Longleftrightarrow\quad C(t) = c + \frac{1}{3}\,e^{3t}.$$

Choosing $c = 0$ we obtain $a_{2,p}(t) = \dfrac{1}{3}\,e^{3t}e^{-4t} = \dfrac{1}{3}\,e^{-t}$

and hence

$$a_2(t) = c\,e^{-4t} + \frac{1}{3}\,e^{-t} \quad \text{and with } a_2(0) = 0 \text{ we get } c = -1/3$$

$$\Longrightarrow a_2(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)$$

$$\Longrightarrow v^{**}(x,t) = a_2(t)\,\sin(2x) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)\sin(2x).$$

d) With the notation from a), b) and c) superposition gives

$$v(x,t) = v^{*}(x,t) + v^{**}(x,t) = \frac{1}{6}\left(2e^{-t} + e^{-4t}\right)\sin(2x)\,\cdot$$

and

$$u(x,t) \;=\; v(x,t) + t \;=\; \frac{1}{6}\left(2e^{-t} + e^{-4t}\right)\sin(2x) \;+\; t\,.$$

Discussion: 23.06.- 26.06.2025