Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Homework sheet 5

Exercise 1:

Using a suitable product ansatz, solve the following Dirichlet boundary value problem for the Laplace equation on the circle $r^2 = x^2 + y^2 \le 9$.

$$\begin{aligned} r^2 u_{rr} + r u_r + u_{\varphi\varphi} &= 0 & \quad 0 \le r < 3 \\ u(3, \varphi) &= \cos^2(\varphi) & \quad \varphi \in \mathbb{R}. \end{aligned}$$

Hints:

- See lecture pages 85-88.
- To solve Euler's equation
  $r^2 \cdot w''(r) + ar \cdot w'(r) + b \cdot w(r) = 0$ use the ansatz $w(r) = r^k$.
- It holds: $\cos^2(\varphi) = \dfrac{1}{2}(1 + \cos(2\varphi))$.

Solution :

By inserting the product ansatz: $u = w(r) \cdot v(\varphi)$ into the Laplace equation in polar coordinates we obtain

$$r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0$$

$$r^2 w'' v + r w' v + w v'' = 0 \implies \frac{r^2 w'' + r w'}{w} = -\frac{v''}{v} = \lambda$$

The solutions of $v''/v = -\lambda$ depend on the sign of λ (see lecture page 88), but only $2\pi-$periodic solutions are possible here:

$$v_k(\varphi) = c_1 \cos(k\varphi) + c_2 \sin(k\varphi), \qquad \lambda = k^2 \quad k \in \mathbb{N}_0$$

For w we have to solve the (Euler's) differential equation

$$r^2 w'' + r w' - k^2 w = 0$$

$\underline{k = 0:}$ $\quad r^2 w''(r) + r w'(r) = 0$ $\quad$ we obtain for $g := w'$

$$r g'(r) + g(r) = 0 \iff r \cdot \frac{dg}{dr} = -g \iff \frac{dg}{g} = -\frac{dr}{r}$$

$$w' = g = \frac{d_0}{r} \implies \boxed{w_0 = a_0 + b_0 \ln(r) .}$$

$\underline{k \neq 0:}$ Euler ODE.: $\quad r^2 w''(r) + r w'(r) - k^2 w = 0$

Substitution $r = e^t$ or the ansatz $w(r) = r^\gamma$ gives

$-k^2 \cdot r^\gamma + r \cdot \gamma \cdot r^{\gamma-1} + r^2 \cdot \gamma \cdot (\gamma - 1) \cdot r^{\gamma-2} = 0$

$\iff r^\gamma \left(-k^2 + \gamma + \gamma^2 - \gamma\right) = 0 \iff \gamma = \pm k$

and hence $\boxed{w_k(r) = a_k r^{-k} + b_k r^k}$

Since the solution should be defined in a circle around zero, i.e. it should remain bounded, the negative powers and the $\ln -$ terms are not suitable here.

Altogether we get the solutions:

$u_0(r, \phi) = c_0, \qquad u_k(r, \phi) = (c_k \cos(k\varphi) + d_k \sin(k\varphi))\, r^k,\ k \in \mathbb{N}\,.$

Each linear combination of these solutions is again a solution of the PDE:

$$u(r, \varphi) = a_0 + \sum_{k=1}^{m} (c_k \cos(k\varphi) + d_k \sin(k\varphi))\, r^k\,.$$

The boundary condition reads

$$\begin{aligned} u(3, \varphi) &= a_0 + \sum_{k=1}^{m} (c_k \cos(k\varphi) + d_k \sin(k\varphi))\, 3^k \\ &= \cos^2(\varphi) = \frac{1}{2}\left(1 + \cos(2\varphi)\right)\,. \end{aligned}$$

A comparison of coefficients results in

$$u(r, \varphi) = \frac{1}{2} + \frac{r^2}{18} \cos(2\varphi)\,.$$

Exercise 2:

a) Using a product ansatz, derive a series representation for the solution of the following Neumann problem.

$$\begin{array}{rcll} u_t &=& u_{xx}, & 0<x<1,\, t>0, \\ u(x,0) &=& g(x), & 0<x<1, \\ u_x(0,t) &=& u_x(1,t)=0 & t>0. \end{array}$$

b) Solve the initial boundary value problem a) with $g(x) = 3 + 4\cos(2\pi x)$.

Solution:

a) The ansatz $u(x,t) = v(x)\cdot w(t)$ yields:

$v'' = -\lambda\, v, \quad \dot{w} = -\lambda w, \quad v'(0) = v'(1) = 0\,.$

Case distinction under the condition that the solution does not vanish:

$$\begin{aligned} \lambda = 0 &\Longrightarrow v(x) = a_0 + b_0 x, \quad v' = b_0 = 0 \\ &\Longrightarrow v_0(x) = a_0\,. \\ \lambda < 0 &\Longrightarrow v(x) = a e^{\sqrt{-\lambda}x} + b e^{-\sqrt{-\lambda}x} \\ &\qquad v'(0) = 0 \Longleftrightarrow a = b \\ &\qquad v'(1) = 0 \Longleftrightarrow a\sqrt{-\lambda}(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}) = 0 \\ &\qquad \Longleftrightarrow (u\equiv 0) \vee (e^{\sqrt{\lambda}} = e^{-\sqrt{\lambda}} \Longleftrightarrow \lambda = 0) \quad \text{Contradiction!} \\ \lambda > 0 &\Longrightarrow v(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \\ &\qquad v'(x) = (\sqrt{\lambda})(-a\sin(\sqrt{\lambda}\,x) + b\cos(\sqrt{\lambda}\,x) \\ &\qquad v'(0) = 0 \Longleftrightarrow b = 0 \\ &\qquad v'(1) = 0 \Longleftrightarrow (u\equiv 0) \vee (\sin(\sqrt{\lambda}) = 0 \Longleftrightarrow \lambda_k \;=\; k^2\pi^2). \end{aligned}$$

So overall we get

$$v_k(x) = \cos(k\pi x), \quad k\in\mathbb{N}_0\,.$$

One can easily calculate for the time component

$$w_k(t) = e^{-k^2\pi^2 t}, \quad k\in\mathbb{N}_0\,.$$

Hence as a series representation for the solution we obtain

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t}\cos(k\pi x)\,.$$

To fulfill:

$$u(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x)\,.$$

To determine the coefficients for general data, the data is continued evenly and $2-$ periodically and the Fourier coefficients are determined

$$a_k = 2\int_0^1 g(x)\cos(k\pi x)\,dx\,.$$

More on this on sheets 6.

b) For $g(x) = 1 + \cos(2\pi x)$ one reads from the condition

$$u(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x) \stackrel{!}{=} 3 + 4\cos(2\pi x)$$

directly that with the choice $a_0 = 6,\ a_2 = 4,\ a_k = 0$ otherwise,
the solution

$$u(x,t) = \frac{a_0}{2} + a_2 e^{-2^2\pi^2 t}\cos(2\pi x) = 3 + 4e^{-4\pi^2 t}\cos(2\pi x)$$

is obtained for the initial boundary value problem.

Submission deadline: 13.06.2025