Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
Prof. Dr. J. Struckmeier
Dr. H. P. Kiani

Differential Equations II for Engineering Students

Work sheet 4

Exercise 1: See Lecture pages 47-53

Consider the initial value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0 \quad \text{for } x \in \mathbb{R},\, t \in \mathbb{R}^+ \\ u(x,0) &= 0 \quad \text{for } x \in \mathbb{R}, \\ u_t(x,0) &= 2xe^{-x^2} \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

a) Rewrite the PDE in matrix form.

b) Carry out the substitution $\alpha = x + \frac{t}{4}$, $\mu = x - t$ and give the PDE in matrix notation for $v(\alpha\,,\mu) := u(x,t)$.

c) Solve the PDE for u by first solving the PDE for $\boldsymbol{v}$ and transforming back afterwards.

d) Determine the solution u for the initial value problem.

Solution:

a) Matrix form: $(\nabla^T \boldsymbol{A}\, \nabla)u + (b^T\nabla)u + cu = h\,.$

Here the Matrix form of the PDE is

$$(\nabla^T A \nabla)u \;=\; \nabla^T \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & -4 \end{pmatrix} \nabla \cdot u = \mathbf{0}$$

b) With $S^T := \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & -1 \end{pmatrix}$ we obtain the following PDE for v

$\nabla^T_{\alpha\mu}\, \boldsymbol{S}^T \boldsymbol{A}\, \boldsymbol{S}\, \nabla_{\alpha\mu}\, v = \mathbf{0}.$

where $\boldsymbol{S}^T \boldsymbol{A}\, \boldsymbol{S} = \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{25}{8} \\ \frac{25}{8} & 0 \end{pmatrix}$

Hence the PDE for v is

$$\nabla^T_{\alpha\mu} \boldsymbol{S}^T \boldsymbol{A}\, \boldsymbol{S}\, \nabla_{\alpha\mu}\, v = \left(\frac{\partial}{\partial\alpha},\, \frac{\partial}{\partial\mu}\right) \begin{pmatrix} 0 & \frac{25}{8} \\ \frac{25}{8} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\alpha} \\ \frac{\partial}{\partial\mu} \end{pmatrix} v = \frac{25}{4} v_{\alpha\mu} = 0 \Longleftrightarrow \boxed{v_{\alpha\mu} = 0}$$

c) Solution of the transformed PDE:

From $(v_\alpha)_\mu = 0$ it follows that v_α does not depend on μ.

$$v(\alpha,\mu)_\alpha = \phi(\alpha) \overset{\int d\alpha}{\Longrightarrow} v(\alpha,\mu) = \Phi(\alpha) + \chi(\mu)$$

and

$$u(x,t) = v(\alpha,\mu) = \Phi(x + \frac{t}{4}) + \chi(x-t)$$

with sufficiently smooth functions Φ and χ.

d) From the initial data we obtain two conditions

$$u(x,0) = \Phi(x) + \chi(x) \overset{!}{=} 0 \quad \text{and}$$
$$u_t(x,0) = \frac{1}{4}\Phi'(x) - \chi'(x) \overset{!}{=} 2xe^{-x^2}.$$

From the first equation we obtain

$$\chi(x) = -\Phi(x)$$

and thus the second equation reads

$$\frac{1}{4}\Phi'(x) + \Phi'(x) \overset{!}{=} 2xe^{-x^2} \iff \Phi'(x) = \frac{4}{5}\left(2xe^{-x^2}\right).$$

Integration delivers

$$-\chi(x) = \Phi(x)) = -\frac{4}{5}e^{-x^2}.$$

The solution to the initial value problem is therefore given by

$$u(x,t) = \Phi(x + \frac{t}{4}) + \chi(x-t) = -\frac{4}{5}e^{-(x+\frac{t}{4})^2} + \frac{4}{5}e^{-(x-t)^2}.$$

Exercise 2: Hint: See lecture page 60 and 65.

a) Let α be a fixed real number from $\mathbb{R}\setminus\{0\}$. For which real-valued functions $g:\mathbb{R}\to\mathbb{R}$ are the following functions harmonic in $\mathbb{R}^2$?

i) $\tilde{u}(x,y) = \cos(\alpha x)\cdot g(y)$,   ii) $u(x,y) = \frac{1}{2}\cdot(x^3 + g(x)\cdot y^2)$.

b) Let $\Omega := \left\{(x,y)^T \in \mathbb{R}^2 : x^2+y^2 < 16\right\}$ and u be the solution of the boundary value problem

$$\Delta u(x,y) = 0 \qquad \text{in } \Omega, \qquad u(x,y) = \frac{2y^2}{x^2+y^2} \qquad \text{on} \quad \partial\Omega.$$

Determine the value of u in the origin.

- Use polar coordinates and the mean value property (lecture page 65).
- Note: $\sin^2(\varphi) = \dfrac{1-\cos(2\varphi)}{2}$.

Solution:

a) i) $\Delta\tilde{u}(x,y) = -\alpha^2\cos(\alpha x)g(y) + \cos(\alpha x)\cdot g''(y) \stackrel{!}{=} 0, \qquad \forall x,\, y \in \mathbb{R}$

$\Longrightarrow g''(y) - \alpha^2 g(y) \stackrel{!}{=} 0$.

This is an ODE with characteristic polynomial $P(\lambda) = \lambda^2 - \alpha^2$

and the general solution $g(y) = k_1 e^{-\alpha y} + k_2 e^{\alpha y}$.

$\tilde{u}$ is harmonic in $\mathbb{R}^2$ if and only if

$$g(y) = k_1 e^{-\alpha y} + k_2 e^{-\alpha y} \qquad k_1,\, k_2 \in \mathbb{R}.$$

ii) $\Delta u(x,y) = \Delta\left(\frac{1}{2}\cdot(x^3 + g(x)\cdot y^2)\right) \stackrel{!}{=} 0, \qquad \forall, x,\, y \in \mathbb{R}$

$\Longrightarrow \frac{1}{2}\cdot(6x + g''(x)y^2 + 2g(x)) \stackrel{!}{=} 0, \qquad \forall, x,\, y \in \mathbb{R}$

$\Longrightarrow g''(x) = 0$ and $6x + 2g(x) = 0 \Longrightarrow g(x) = -3x$.

$\tilde{u}$ is harmonic in $\mathbb{R}^2$ if and only if $g(x) = -3x$.

b) Let K_4 be the edge of the disk with radius 4 around zero and

$c(t) = (4\cos(t),\, 4\sin(t)), \qquad t \in [0, 2\pi]$

a parameterization of K_4. Using the mean value theorem one obtains

$$u(0,0) = \frac{1}{2\pi\cdot 4}\int_{K_4}\frac{2y^2}{x^2+y^2}\, d(x,y) = \frac{1}{8\pi}\int_0^{2\pi}\frac{2\cdot 16\sin^2(t)}{16\cos^2(t) + 16\sin^2(t)}\cdot \|\dot{c}(t)\|\, dt$$

$$= \frac{1}{8\pi}\int_0^{2\pi}(1-\cos(2t))\cdot 4\, dt = 1.$$

Exercise 3: Hint: Lecture pages 61-64 and 69

a) Let

$$\Omega_2 = \{(x,y)^\top \in \mathbb{R}^2 \ : \ 1 < x^2 + y^2 < 4\}.$$

Determine the solutions of

$$\begin{cases} \Delta u = 0 & \text{on} & \Omega_2, \\ u(x,y) = 1 & \text{for} & x^2+y^2 = 1, \\ u(x,y) = 3 & \text{for} & x^2+y^2 = 4. \end{cases}$$

Is the solution unique?

b) Let

$$\Omega_3 = \{(x,y,z)^\top \in \mathbb{R}^3 \ : \ 1 < x^2 + y^2 + z^2 < 4\}.$$

Determine the solutions of

$$\begin{cases} \Delta u = 0 & \text{on} & \Omega_3, \\ u(x,y,z) = 1 & \text{for} & x^2+y^2+z^2 = 1, \\ u(x,y,z) = 3 & \text{for} & x^2+y^2+z^2 = 4. \end{cases}$$

Is the solution unique?

Solution:

According to lecture pages 63/64, every rotationally symmetrical harmonic function on $\mathbb{R}^n \setminus \{0\}$ can be represented as

$$u(\boldsymbol{x}) = a\Phi(\boldsymbol{x}) + c, \qquad a,\, c \in \mathbb{R}$$

by using the fundamental solution $\Phi(\boldsymbol{x})$.

(a) Since $(0,0)^\top \notin \Omega_2$, using

$$\Phi(x,y) = -\frac{1}{2\pi} \ln(\|(x,y)\|_2)$$

we obtain

$$u(x,y) = -\frac{a}{2\pi} \ln\left(\sqrt{x^2+y^2}\right) + +c.$$

The boundary values require

$$\begin{aligned} u(x,y) = 1 \quad & \text{for} \quad x^2+y^2 = 1, \\ u(x,y) = 3 \quad & \text{for} \quad x^2+y^2 = 4. \end{aligned}$$

From the first boundary value we obtain

$-\frac{a}{2\pi} \ln(1) + c = c \overset{!}{=} 1$

and therefore the second boundary value gives

$-\frac{a}{2\pi} \ln(2) + 1 \overset{!}{=} 3 \Rightarrow a = -\frac{4\pi}{\ln(2)}$,

i.e.,

$$u(x,y) = \frac{2}{\ln(2)} \ln\left(\sqrt{x^2+y^2}\right) + 1.$$

The uniqueness follows from the maximum principle (see page 69 of the lecture).

- Analogous to part a), the lecture provides

$$u(x, y, z) = a\Phi(x, y, z) + c, \qquad a, c \in \mathbb{R},$$

with fundamental solution

$$\Phi(x, y, z) = -\frac{1}{4\pi}\|(x, y, z)\|_2^{-1}.$$

The boundary values give the conditions

$$\frac{a}{4\pi} + c = 1, \qquad \frac{a}{8\pi} + c = 3 \qquad \Rightarrow \qquad a = -16\pi, \quad c = 5,$$

i.e.

$$u(x, y, z) = -\frac{4}{\sqrt{x^2 + y^2 + z^2}} + 5.$$

The uniqueness again follows from the maximum principle.

Discussion: 09.06.- 13.06.2025