Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Homework sheet 4

Exercise 1: See lecture page 52.

Determine the type of the following partial differential equations

a) $2u_{xx} - 8u_{xy} + 8u_{yy} + u_y = u\,,$

b) $2u_{xy} + u_{yy} + xu_x = \cos(y)\,,$

c) $3u_{xx} + 2u_{xy} + u_{yy} = 0\,,$

d) $u_{xx} + e^x u_{yy} + \sin(x)(u_x + u_y) = y + x\,,$

e) $(x^2 + y^2)u_{xx} + 2(x + y)u_{xy} + u_{yy} = 0\,.$

Solution :

a) $2u_{xx} - 8u_{xy} + 8u_{yy} + u_y = u$

$$2 \cdot 8 - 4^2 = 0 \qquad \text{parabolic}\,.$$

b) $2u_{xy} + u_{yy} + xu_x = \cos(y)$

$$1 \cdot 0 - 1 = -1 \qquad \text{hyperbolic}\,.$$

c) $3u_{xx} + 2u_{xy} + u_{yy} = 0$

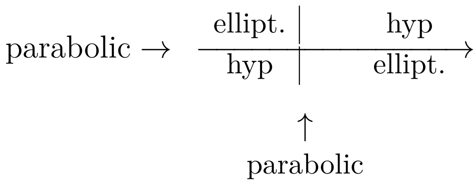
$$3 \cdot 1 - 1^2 = 2 \qquad \text{elliptic}\,.$$

d) $u_{xx} + e^x u_{yy} + \ldots = \ldots$

$$1 \cdot e^x - 0^2 > 0 \qquad \text{elliptic}\,.$$

e) $(x^2+y^2)u_{xx} + 2(x+y)u_{xy} + u_{yy} = 0$

$$x^2+y^2-(x+y)^2 = -2xy \qquad \begin{cases} \text{parabolic for} & xy = 0, \\ \text{hyperbolic for} & xy > 0, \\ \text{elliptic for} & xy < 0. \end{cases}$$



Exercise 2: See lecture pages 59-70.

Let u be a harmonic function in $\Omega := \{\binom{x}{y} \in \mathbb{R}^2 : x^2 + y^2 < 4\}$. Determine the value of $u(0,0)$ for

a) $u(x,y) = \dfrac{x+y+1}{4}$ on the boundary of $\Omega = \partial\Omega$ using the Poisson integral representation of the solution.

b) $u(x,y) = x^2y + 2$ on $\partial\Omega$, using the mean value property of harmonic functions.

c) $u(x,y) = x^2 - y^2$ on $\partial\Omega$, using the uniqueness property of the solution.

d) $u(x,y) = x^2+y^2$ on $\partial\Omega$, without calculation, using the maximum/minimum principle.

Solution:

a) $u(x,y) = \dfrac{x+y+1}{4}$ on the boundary of $\Omega = \partial\Omega$ using the Poisson integral representation of the solution.

Let K_2 be the circle of radius $R = 2$ and $c(t) = (2\cos\phi,\, 2\sin\phi)$ a parametrization of K_2. Then, from Poisson's integral formula it follows

$$u(x,y) = \frac{R^2 - x^2 - y^2}{2\pi R} \int_{\|z\|=R} \frac{g(\boldsymbol{z})}{\|\boldsymbol{z} - \boldsymbol{x}\|^2}\, d\boldsymbol{z}$$

$$u(0,0) = \frac{4}{4\pi} \int_{\|z\|=2} \frac{z_1 + z_2 + 1}{16}\, d(z_1,z_2) = \frac{1}{4\pi} \int_0^{2\pi} \frac{2\cos\phi + 2\sin\phi + 1}{4} \cdot 2\, dt = \frac{1}{4}\,.$$

b) $u(x,y) = x^2y + 2$ on $\partial\Omega$, using the mean value property of harmonic functions.

Let K_2 and $c(t)$ be defined as in part a). Then from the mean value property it follows

$$\begin{aligned} u(0,0) =& \frac{1}{2\pi \cdot 2} \int_{K_2} (x^2y + 2)\, d(x,y) = \frac{1}{4\pi} \int_0^{2\pi} (4\cos^2(\phi) 2\sin(\phi) + 2) \cdot 2\, dt \\ =& \frac{1}{2\pi} \left[-\frac{8\cos^3(\phi)}{3} + 2t \right]_0^{2\pi} = 2. \end{aligned}$$

c) $u(x,y) = x^2 - y^2$ on $\partial\Omega$, using the uniqueness property of the solution.

$u(x,y) = x^2 - y^2$ solves the potential equation in the whole disk, so it is the unique solution. And thus $u(0,0) = 0$.

d) $u(x,y) = x^2+y^2$ on $\partial\Omega$, without calculation, using the maximum/minimum principle.

$u(x,y)$ is constant on the boundary of Ω. Since the maximum and minimum value of u are assumed on the boundary u is constant on the entire circular disc and $u(0,0) = 4$.

Submission deadline: 13.06.2025