

## Differential Equations II for Engineering Students

### Work sheet 3

#### Exercise 1:

a) Determine the entropy solution  $u(x, t)$  to the Burgers' equation  $u_t + uu_x = 0$  for the

$$\text{initial values } u(x, 0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 2 & 1 < x \end{cases}$$

b) Given the following initial value problem for  $u(x, t)$ :

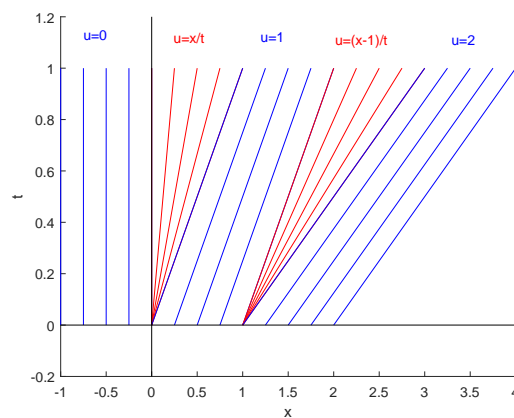
$$u_t + u \cdot u_x = 0, \quad x \in \mathbb{R}, t \in \mathbb{R}^+$$

$$u(x, 0) = \begin{cases} \frac{1}{2} & x \leq 0, \\ 0 & 0 < x \leq 1, \\ -2 & 1 < x. \end{cases}$$

- (i) Compute the weak solution for  $t \in [0, \tilde{t}]$  with a sufficiently small  $\tilde{t}$ .
- (ii) Up to which value  $t^*$  can the solution to (i) be maximally continued?
- (iii) Provide the weak solution for  $t > t^*$ .

#### Solution sketch:

$$\text{a) } u_t + uu_x = 0, \quad u(x, 0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 2 & x > 1 \end{cases} \quad \longrightarrow \quad \text{Two rarefaction waves}$$



$$u(x, t) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t} & 0 \leq x \leq t \\ 1 & t \leq x \leq t+1 \\ \frac{x-1}{t} & t+1 \leq x \leq 2t+1 \\ 2 & x \geq 2t+1 \end{cases}$$

- b) (i) At two jump points of the initial data, we introduce two shock waves.

The jump condition requires:

$$\dot{s}_1(t) = \frac{\frac{1}{2} + 0}{2} = \frac{1}{4} \quad \text{and} \quad \dot{s}_2(t) = \frac{0 - 2}{2} = -1.$$

We obtain the shock fronts

$$s_1(t) = \frac{1}{4}t \quad \text{and} \quad s_2(t) = 1 - t.$$

For sufficiently small  $t$

$$u(x, t) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{4}t, \\ 0 & \frac{1}{4}t < x \leq 1 - t, \\ -2 & 1 - t < x. \end{cases}$$

is a weak solution.

- (ii) For  $t^*$  with

$$\frac{1}{4}t^* = 1 - t^* \iff \frac{5}{4}t^* = 1 \iff t^* = \frac{4}{5}$$

the shock fronts meet and the solution from a) becomes ambiguous.

- (iii) For  $t^* = \frac{4}{5}$  it holds  $s_1(t) = s_2(t) = \frac{1}{5}$  and

$$u(x, \frac{4}{5}) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{5}, \\ -2 & x > \frac{1}{5}. \end{cases}$$

We add the new shock front

$$s_3(t) = \frac{1}{5} + \dot{s}_3(t - \frac{4}{5}) = \frac{1}{5} + \frac{\frac{1}{2} - 2}{2}(t - \frac{4}{5})$$

and obtain for  $t > \frac{4}{5}$

$$u(x, t) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{5} - \frac{3}{4}(t - \frac{4}{5}) = \frac{4}{5} - \frac{3}{4}t, \\ -2 & x > \frac{4}{5} - \frac{3}{4}t. \end{cases}$$

**Exercise 2:**

Determine entropy solutions to the differential equation

$$u_t + (f(u))_x = 0$$

with the flow function  $f(u) = \frac{(u-2)^4}{2}$  and initial conditions

$$\text{a) } u(x, 0) = \begin{cases} 2 & x \leq 0, \\ 1 & 0 < x, \end{cases} \quad \text{and} \quad \text{b) } u(x, 0) = \begin{cases} 1 & x \leq 0, \\ 2 & 0 < x. \end{cases}$$

Note: Only solutions for the given initial values are required. You don't need to give solutions for general initial values!

**Solution:**

Using usual notation we have  $f(u) = \frac{(u-2)^4}{2}$ . [1 point]

On the characteristic curves it holds

$$\dot{x}(t) = f'(u) = 2(u-2)^3 \quad \text{and} \quad \dot{u}(t) = 0.$$

The characteristics are straight lines with the constant slope  $2(u(x(0), 0) - 2)^3$ .

In part a), an ambiguity of the solution obtained using the methods of characteristics arises immediately (i.e. already at  $t = 0$ ). A shock front  $s(t)$  must be introduced with  $u_l = 2$  and  $u_r = 1$  [1 point]

with

$$\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{(2-2)^4}{2} - \frac{(1-2)^4}{2}}{2 - 1} = -\frac{1}{2} \quad [1 \text{ point}]$$

We obtain

$$u(x, t) = \begin{cases} u_l = 2 & x < s(t) = -\frac{t}{2} \\ u_r = 1 & -\frac{t}{2} < x. \end{cases} \quad [1 \text{ point}]$$

For part b) the method of characteristics gives

$$u(x, t) = \begin{cases} 1 & x \leq x_0 + f'(u_l)t = 0 + 2(1-2)^3t = -2t, \\ ? & -2t \leq x \leq 0, \\ 2 & x \geq x_0 + f'(u_r)t = 0 + 2(2-2)^3t = 0. \end{cases}$$

A rarefaction wave must therefore be introduced. [1 point]

With

$$f'(u) = 2(u-2)^3 = v \implies g(v) := (f')^{-1}(v) = \left(\frac{v}{2}\right)^{\frac{1}{3}} + 2$$

we have the solution

$$u(x, t) = \begin{cases} 1 & x \leq -2t, \\ g\left(\frac{x}{t}\right) = \left(\frac{x}{2t}\right)^{\frac{1}{3}} + 2 & -2t \leq x \leq 0 \\ 2 & x \geq 0. \end{cases} \quad [2 \text{ points}]$$

**Exercise 3:** (Only for the very fast students)

Physical processes described by smooth solutions to hyperbolic differential equations are generally reversible. If the solution is known at a certain time, one can use it to determine the solution at both later and earlier times.

Draw the characteristics for both initial value problems for the Burgers' equation  $u_t + uu_x = 0$  with initial data

$$u_1(x, 0) = \begin{cases} 1 & x \leq 0, \\ 0 & x > 0. \end{cases}$$

and

$$u_2(x, 0) = \begin{cases} 1 & x < -\frac{1}{4}, \\ \frac{1}{2} - 2x & -\frac{1}{4} \leq x \leq \frac{1}{4}, \\ 0 & x > \frac{1}{4}. \end{cases}$$

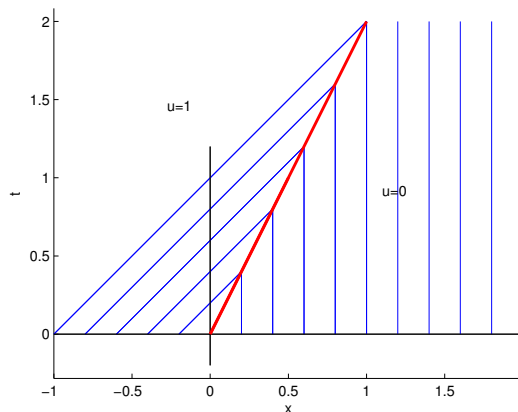
Determine the solution  $u(x, 1)$  for both initial value problems at time  $t = 1$ .

What do you conclude from your results regarding reversibility of non-smooth solutions to Burgers' equation?

**Solution sketch to exercise 3:**  $u_t + uu_x = 0$ 

The initial data  $u_1(x, 0) = \begin{cases} 1 & x \leq 0 \\ 0 & x > 0 \end{cases}$  already known  
from the lecture

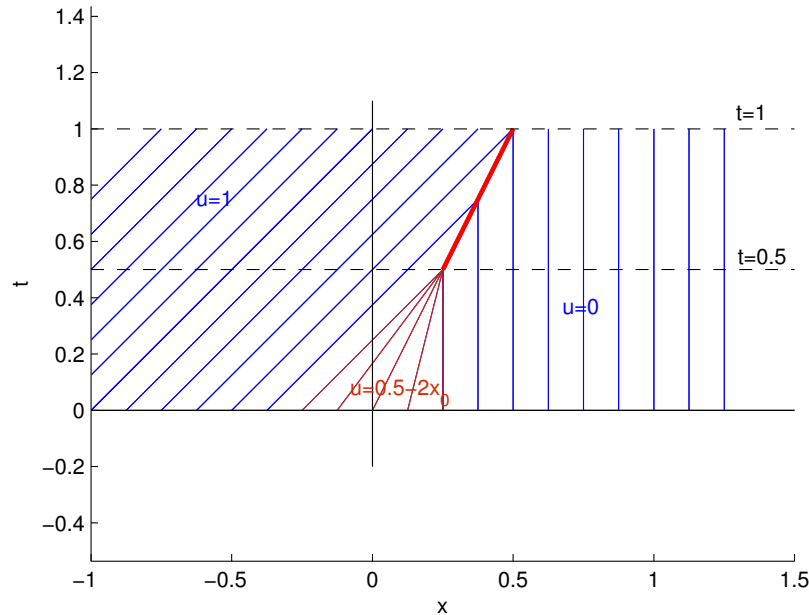
results in the following plot of characteristics



From the very beginning, a shock wave with the velocity  $\frac{1}{2}$  occurs, and it holds

$$u_1(x, 1) = \begin{cases} 1 & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}.$$

The initial data  $u_2(x, 0) = \begin{cases} 1 & x < -\frac{1}{4} \\ \frac{1}{2} - 2x & -\frac{1}{4} \leq x \leq \frac{1}{4} \\ 0 & x > \frac{1}{4} \end{cases}$  gives:



On the characteristics it holds  $x - x_0 = u_0 \cdot t$ .

For  $-\frac{1}{4} \leq x_0 \leq \frac{1}{4}$  and  $t = \frac{1}{2}$  so

$$x\left(\frac{1}{2}\right) = x_0 + \left(\frac{1}{2} - 2x_0\right) \frac{1}{2} = \frac{1}{4}$$

The characteristics that start at  $t = 0$  between  $x = -\frac{1}{4}$  and  $x = \frac{1}{4}$  all intersect at the point  $\left(\frac{1}{4}, \frac{1}{2}\right)$ .

For  $t = \frac{1}{2}$  it holds  $u(x, \frac{1}{2}) = \begin{cases} 1 & x \leq \frac{1}{4} \\ 0 & x > \frac{1}{4} \end{cases}$ . A shock wave is thus created with  $x_0 = \frac{1}{4}$

and  $u_l = 1, u_r = 0$ . Hence for the shock front it holds

$$s\left(\frac{1}{2}\right) = \frac{1}{4}, \quad \dot{s}(t) = \frac{f_l - f_r}{u_l - u_r} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \frac{1}{2}$$

$$\implies s(t) = \frac{1}{4} + \frac{1}{2}\left(t - \frac{1}{2}\right) = \frac{t}{2}.$$

Hence for  $t \geq \frac{1}{2}$  we have:

$$u_2(x, t) = \begin{cases} 1 & x \leq \frac{t}{2}, \\ 0 & x > \frac{t}{2} \end{cases}$$

and then for  $t = 1$  also:

$$u_2(x, 1) = \begin{cases} 1 & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}.$$

Obviously, knowing the solution at time  $t = 1$ , one cannot reconstruct the solution at earlier times around  $t = 0$ .

**Additionally (not required by the assignment):**

For  $t < 0.5$  it holds:

$$u(x, t) = 1 \text{ for } x \leq t - 0.25,$$

$$u(x, t) = 0 \text{ for } x \geq 0.25.$$

In between, i.e. for  $t - 0.25 < x < 0.25$ , the following holds:

$$\frac{dx}{dt} = u, \quad \frac{du}{dt} = 0. \text{ i.e. } u \text{ is constant along characteristics and}$$

$$x = ut + c \iff c = x(0) = x - ut$$

$$\implies u(x, t) = u_0(x - ut) = \frac{1}{2} - 2(x - ut) \iff u \cdot (1 - 2t) = \frac{1}{2} - 2x$$

$$\text{So } u(x, t) = \frac{1 - 4x}{2 - 4t} \text{ für } t - 0.25 < x < 0.25, 0 < t < 0.5.$$

**Discussion: 19.05.-22.05.2025**