Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Homework sheet 3

Exercise 1:

a) Consider the following initial-value problem for $u(x,t)$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$

$$\begin{aligned} u_t + u \cdot u_x &= 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+ \\ u(x,0) &= g(x), \qquad x \in \mathbb{R}\,. \end{aligned}$$

Here let $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonically increasing function with two points of discontinuity (jump points).

For each of the following statements, determine if it is true or false.

(i) There is a unique weak solution.

(ii) In order to obtain the entropy solution, one has to introduce two shock waves.

(iii) The entropy solution is valid for all times, i.e. for any $t \in \mathbb{R}^+$.

Justify your answers.

b) What is the jump condition for the weak solution to

$$u_t + (u^3)_x = 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} 4 & \text{for } x \le 0, \\ 2 & \text{for } x > 0\,? \end{cases}$$

Solution sketch for Exercise 1:

a) (i) The statement is false. Only the entropy condition ensures the uniqueness.

(ii) False. Since the initial data increases monotonically, the entropy solution has no discontinuities.

(iii) The statement is true. One introduces two rarefaction waves that do not get in each other's way.

b) With $f(u) = u^3$ the jump condition for a shock front is:

$$\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{4^3 - 2^3}{4 - 2} = 2 \cdot 16 - 4 = 28\,.$$

Exercise 2: Determine the entropy solution to the Burgers' equation $u_t + uu_x = 0$ with the initial data

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

at the time $t = 2$. What new problem occurs at $t = 2$?

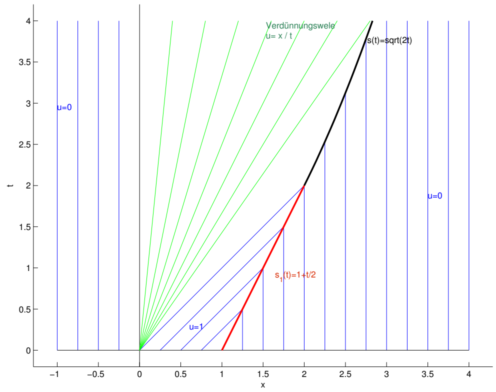
Additional voluntary task: Determine the solution for $t > 2$.

Solution: $u_t + uu_x = 0$

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 0 & x > 1 \end{cases}$$

It is clear that

- the solution is constant along the characteristic lines
- the characteristics are straight lines with the slope $1/u_0$ in the $(x,t)-$ plane



So first of all, we get

$\dot{s}(t) = \dfrac{1+0}{2}$, $s(t) = 1 + \frac{t}{2}$:

$$u(x,t) = \begin{cases} 0 & x \le 0 \\ \frac{x}{t} & 0 \le x \le t \\ 1 & t \le x \le 1 + \frac{t}{2} \\ 0 & x > 1 + \frac{t}{2}. \end{cases}$$

This solution is valid until t^* with $t^* = 1 + \frac{t^*}{2}$, i.e. $t^* = 2$. At time $t^* = 2$ the rarefaction wave meets the shock wave. For $t \ge 2$ holds for the discontinuity with

$u_l = \dfrac{x}{t}, \qquad u_r = 0$ and $x = s(t)$ on the discontinuity curve

$\dot{s}(t) = \dfrac{\frac{s(t)}{t} + 0}{2} = \dfrac{s(t)}{2t}$ This is an ordinary differential equation for $s(t)$.

$$\left.\begin{array}{l} \dfrac{ds}{s} = \dfrac{dt}{2t} \qquad \Longrightarrow \quad s = c\sqrt{t} \\ s(2) = 2 \end{array}\right\} \quad \Longrightarrow \qquad c = \sqrt{2}, \;\; s = \sqrt{2t}$$

So the discontinuity moves on the curve $x(t) = \sqrt{2t}$.

$$u(x,t) = \begin{cases} 0 & x \leq 0 \\ \frac{x}{t} & 0 \leq x \leq \sqrt{2t} \\ 0 & x > \sqrt{2t}. \end{cases}$$

Exercise 3:

We discuss again the simple traffic flow model from Sheet 1 with the notation introduced there:

$$\begin{aligned} u(x,t) &= \text{ density of vehicles (vehicles/length) at point } x \text{ at time } t\,, \\ v(x,t) &= \text{ velocity at point } x \text{ at time } t\,, \\ q(x,t) &= u(x,t)\cdot v(x,t) = \text{ flow } = \text{ number of vehicles passing } x \text{ at time } t \text{ per time unit.} \end{aligned}$$

We improve our model from Sheet 1 by incorporating maximal density and a maximal velocity

$$\begin{aligned} u_{max} &= \text{ maximal density of vehicles (bumper to bumper),} \\ v_{max} &= \text{ maximal velocity} \end{aligned}$$

This can be done, for example, as follows:

$$v(x,t) := v(u(x,t)) := v_{max}\left(1 - \frac{u(x,t)}{u_{max}}\right)$$

a) Set up the continuity equation ($u_t + q_x = 0$).

b) Show again that the characteristics are straight lines and determine their slopes.

c) Sketch the characteristics for

$$\begin{aligned} v_{max} &= 1 \quad \text{(Here it has been scaled appropriately!)} \\ u(x,0) &= \begin{cases} u_l = u_{max}/2 & x < 0 \\ u_r = u_{max} & x > 0 \quad \text{(red traffic light/traffic jam etc.)} \end{cases} \end{aligned}$$

d) For the Burgers' equation we allowed shock waves only in the case $u_l > u_r$. There must obviously be a different condition here. What could be the reason for that?

Note: This question can not be answered completely only with help of the lecture slides. You can only make a guess here!

Solution hint to Exercise 3:

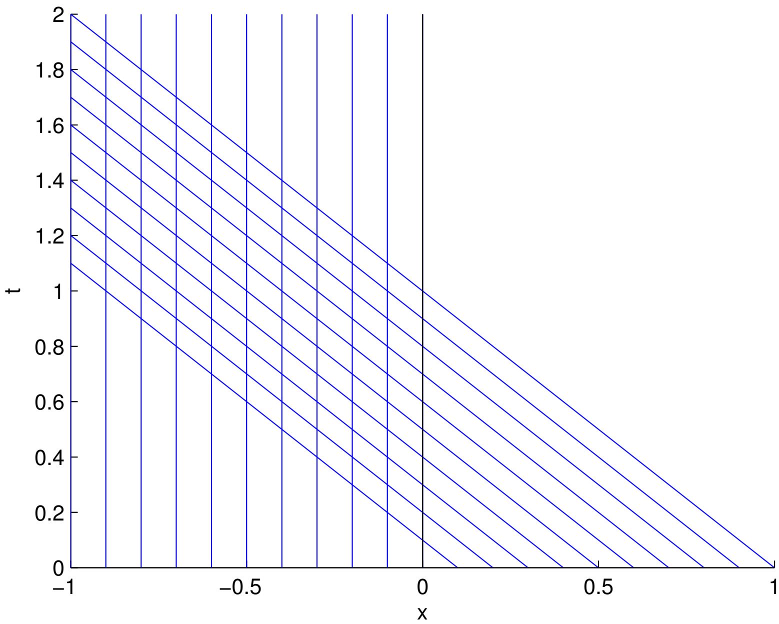
a) $$u_t + \left(v_{max}\, u\,(1 - \frac{u}{u_{max}})\right)_x = u_t + \left(v_{max}\,(u - \frac{u^2}{u_{max}})\right)_x = u_t + (v_{max}\,(1 - \frac{2u}{u_{max}}))u_x = 0$$

b) On the characteristic $x(t)$ the following applies:

$\dot{x}(t) = \left(v_{max}\,(1 - \frac{2u}{u_{max}})\right)$ and $\dot{u}(t) = 0$. The characteristic through a point $(x(0),0)$ has the constant slope in the x–t– plane as a straight line $\left(v_{max}\,(1 - \frac{2u(x(0),0)}{u_{max}}\right)^{-1}$.

The characteristics are straight lines again.

c) Sketch of characteristics:



d) The entropy condition from the lecture is only for convex flow functions f (here q). Since f' is monotonically decreasing, the entropy condition from the lecture does not apply in our case.

What still applies is the graphic interpretation: No information comes out of the shock wave!! So

$$f'(u_l) > \dot{s} > f'(u_r)$$

Since f' is monotonically decreasing, we have the condition for shock waves $u_l < u_r$.

Submission deadline: 23.05.2025