Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Work sheet 2

Exercise 1

Compute the solutions of the following initial value problems for $u(x,t)$:

a)

$$\begin{aligned} u_t + \tfrac{1}{2} u_x &= 0, && x \in \mathbb{R},\, t \in \mathbb{R}^+, \\ u(x,0) &= 2\sin(x), && x \in \mathbb{R}\,. \end{aligned}$$

b) (Exam SuSe17, Ex.2a)

$$\begin{aligned} u_t + \tfrac{1}{2} u_x &= -4\,(u+1), && x \in \mathbb{R},\, t \in \mathbb{R}^+, \\ u(x,0) &= 2\sin(x), && x \in \mathbb{R}\,. \end{aligned}$$

Solution: Using the method of characteristics one computes:

a) $\frac{dx}{dt} = \frac{1}{2} \Longrightarrow x(t) = \frac{t}{2} + \tilde{C} \Longrightarrow 2x - t = C.$

$\frac{du}{dt} = 0 \Longrightarrow u = D.$

General solution:

$D = f(C) \Longrightarrow u = f(2x-t)\,.$

The initial condition requires:

$u(x,0) = f(2x-0) \stackrel{!}{=} 2\,\sin(x) \Longrightarrow f(x) = 2\,\sin(\frac{x}{2})\;.$

Hence, the solution to the IVP is

$$u(x,t) = 2\,\sin\left(x - \frac{t}{2}\right).$$

b) $\frac{dx}{dt} = \frac{1}{2} \Longrightarrow x(t) = \frac{t}{2} + C \Longrightarrow 2x - t = C$

$\frac{du}{dt} = -4(u+1) \Longrightarrow \frac{du}{u+1} == -4\,dt \Longrightarrow \ln(|u+1|) = -4t + d$

$|u+1| = e^{-4t} \cdot \tilde{d}, \qquad \tilde{d} \in \mathbb{R}^+$

$\Longrightarrow u+1 = \tilde{d}e^{-4t}$ or $-u-1 = \tilde{d}e^{-4t}$

$\Longrightarrow u = -1 \pm \tilde{d}e^{-4t}, \qquad \tilde{d} \in \mathbb{R}^+\,.$

Since $u = -1$ is also a solution, we get

$u(x(t),t) = D \cdot e^{-4t} - 1, \qquad D \in \mathbb{R} \qquad$ or $\qquad D = e^{4t}(u+1)\,.$

Applying the implicit function theorem we have the general solution:

$D = f(C) \Longrightarrow e^{4t}(u+1) = f(2x-t) \Longrightarrow u(x,t) = e^{-4t} \cdot f(2x-t) - 1\,.$

The initial condition requires:

$u(x,0) = e^{0} \cdot f(2x-0) - 1 \overset{!}{=} 2\sin(x) \Longrightarrow f(x) = 1 + 2\sin(\frac{x}{2})\,.$

$u(x,t) = e^{-4t}\left[2\sin\left(x - \frac{t}{2}\right) + 1\right] - 1\,.$

Exercise 2:

Determine the solution $u(x,y)$ to the following differential equation

$$xu_x + \frac{y}{2}u_y = u\,,$$

that satisfies the condition $u(1,y) = 1 + y^2\,, \qquad y \in \mathbb{R}\,.$

Solution 2:

a) $xu_x + \frac{y}{2}u_y = u\,, \qquad u(1,y) = 1 + y^2$

With $x \neq 0$ as a parameter one computes for the equation $u_x + \frac{y}{2x}u_y = \frac{u}{x}$

$$\begin{array}{lll}
\frac{dy}{dx} = \frac{y}{2x} & \Longrightarrow & \frac{2dy}{y} = \frac{dx}{x} \\
\Longrightarrow 2\ln|y| = \ln|x| + c & \Longrightarrow & e^{2\ln|y|} = e^{\ln|x|+c} \\
\Longrightarrow y^2 = c_1 \cdot x & \Longrightarrow & c_1 = \frac{y^2}{x} \\
\frac{du}{dx} = \frac{u}{x} & \Longrightarrow & \frac{du}{u} = \frac{dx}{x} \\
\ln|u| = \ln|x| + d & \Longrightarrow & u = c_2 \cdot x \Longrightarrow c_2 = \frac{u}{x} \\
c_2 = f(c_1) & \Longrightarrow & \frac{u}{x} = f(\frac{y^2}{x}) \Longrightarrow u(x,y) = x \cdot f(\frac{y^2}{x})
\end{array}$$

This is the general solution. Now determine f using the given condition: Insert $u(1,y) = 1 + y^2$ into the general solution

$u(x,y) = x \cdot f(\frac{y^2}{x})$

$u(1,y) = 1 \cdot f(\frac{y^2}{1}) = f(y^2) \overset{!}{=} 1 + y^2$

Also $f(\mu) = 1 + \mu$ and thus

$u(x,y) = x \cdot f(\frac{y^2}{x}) = x \cdot (1 + \frac{y^2}{x}) = x + y^2$

One can now see that the solution for all $x \in \mathbb{R}$ satisfies the PDE + condition. So the constraint $x \neq 0$ can be omitted.

ALTERNATIVELY:

Auxiliary problem $\qquad xU_x + \frac{y}{2}U_y + uU_u = 0$

$$\begin{cases} \dot{x} = x & \Longrightarrow \quad x = c_1 e^t \\ \dot{y} = \frac{y}{2} & \Longrightarrow \quad y = c_2 e^{\frac{t}{2}} \\ \dot{u} = u & \Longrightarrow \quad u = c_3 e^t \end{cases}$$

It holds (with suitable constants)

$$\begin{cases} x &= cy^2 \\ u &= dx \end{cases}$$

$$c = \frac{x}{y^2}\,, \qquad d = \frac{u}{x}\,, \qquad \phi\left(\frac{x}{y^2}, \frac{u}{x}\right) = 0$$

Assuming solvability, we have

$$\frac{u}{x} = \psi\left(\frac{x}{y^2}\right) \qquad u = x \cdot \psi\left(\frac{x}{y^2}\right)$$

additionally it should hold that $\qquad u(1, y) = \psi\left(\frac{1}{y^2}\right) = y^2 + 1$

$$\Longrightarrow \psi(\mu) = \frac{1}{\mu} + 1 \iff \psi\left(\frac{x}{y^2}\right) = \frac{y^2}{x} + 1 \Longrightarrow \boxed{u = y^2 + x}$$

Exercise 3: (only for people who compute fast)
Given the following initial value problem

$$u_t + 3u \cdot u_x = 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+$$
$$u(x,0) = \begin{cases} 0 & \forall x \le 0 \\ \frac{1}{3} & \forall x > 0 \end{cases}$$

a) Write down the system of characteristic equations.

b) Are the characteristics straight lines?

c) Draw the characteristics through the points $(x_k, 0) := (k, 0)$ for $k \in \{-3, -2, -1, 0, 1, 2, 3\}$.
Compute the values of the solution along these characteristics.

d) Using parts a)-c), can you obtain the values of $u(x,t)$ in the points $(-1,2), (1,2)$ and $(3,2)$?

Solution:

a) Extended problem $\quad U_t + (3u)U_x + 0 \cdot U_u = 0$ implies:

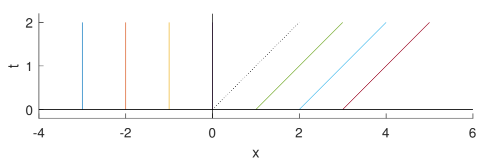
$$\frac{dx}{dt} = 3u, \qquad \frac{du}{dt} = 0 \qquad \Longrightarrow$$
$$u = C, \qquad dx = 3C dt$$
$$\Longrightarrow x(t) = 3Ct + D = 3ut + D \Longrightarrow D = x - 3ut\,.$$

b) The characteristics are straight lines because it holds that
$\frac{du}{dt} = 0 \qquad \Longrightarrow u$ is therefore constant along every characteristic. Also, it holds
$\frac{dx}{dt} = 3u \qquad \Longrightarrow$ so $\frac{dx}{dt}$ is constant along the characteristic, i.e.
the slope of the characteristics is constant. These are straight lines.

c) Sketch

For $x(0) \le 0$ we have $\frac{dx}{dt} = 0$. Hence the characteristics are vertical lines in the $(x,t)-$ plane. On these lines $u = 0$ holds.

For $x(0) > 0$ we obtain $\frac{dx}{dt} = 3 \cdot \frac{1}{3} = 1$. The characteristics are straight lines in the $(x,t)-$ plane with slope one. On these lines $u = \frac{1}{3}$ holds.



d) From the sketch one takes $u(x,t) = 0, \forall x \leq 0$. So in particular, we have $u(-1,2) = 0$.
Furthermore, from the sketch we get $u(x,t) = \frac{1}{3}, \forall x > t$. So we obtain $u(3,2) = \frac{1}{3}$.
The characteristics do not help to determine the solution values $u(x,t)$ for $0 < x < t$, so for example for $u(1,2)$.
The solution to this problem is discussed on the next sheet!

Discussion: 05.05.-08.05.2025