Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Work sheet 1

Exercise 1:

We are looking for solutions of the heat equation in one spatial dimension:

$$u_t - cu_{xx} = 0$$

with a fixed parameter $c \in \mathbb{R}^+$ (thermal conductivity / diffusion coefficient).

Show that for any number $\omega \in \mathbb{R}$ and for any $k \in \mathbb{Z}$

$$u_k(x,t) = \sin(\ k\omega x\ ) \cdot e^{-ck^2\omega^2 t}$$

is a solution of the differential equation .

Obviously, a so-called product ansatz: $u(x,t) = q(t) \cdot p(x)$ leads to solutions of the heat equation.

Solution 1:

Deriving u_k we obtain

$$\begin{aligned}
\frac{\partial}{\partial t} u_k(x,t) &= -ck^2\omega^2 \sin(\ k\omega x\ ) \cdot e^{-ck^2\omega^2 t}, \\
\frac{\partial}{\partial x} u_k(x,t) &= k\omega \cos(\ k\omega x\ ) \cdot e^{-ck^2\omega^2 t}, \\
\frac{\partial^2}{\partial x^2} u_k(x,t) &= -k^2\omega^2 \sin(\ k\omega x\ ) \cdot e^{-ck^2\omega^2 t}.
\end{aligned}$$

Hence

$$\frac{\partial}{\partial t} u_k(x,t) - c\frac{\partial^2}{\partial x^2} u_k(x,t) = 0\,.$$

Remarks:

- According to homework 1), every finite linear combination $\sum_{k=m}^{n} a_k u_k(x,t)$ of these functions is also a solution of the differential equation .
- A so-called product ansatz: $u(x,t) = q(t) \cdot p(x)$ yields

$$p(x)\dot{q}(t) - cp''(x)q(t) = 0.$$

Rearranging provides

$$\frac{\dot{q}(t)}{q(t)} \overset{!}{=} c\frac{p''(x)}{p(x)}$$

Since the right-hand side depends only on x and the left-hand side only on t, both sides must be constant. For example, with a fixed number $\lambda \in \mathbb{R}$:

$$\frac{\dot{q}(t)}{q(t)} \overset{!}{=} c\frac{p''(x)}{p(x)} =: -\lambda.$$

We therefore obtain a system of two ordinary differential equations coupled via the parameter λ. Later in the semester, we will deal with the solution of this system in detail.

Exercise 2:

We now consider the Telegraph Equation.

A signal of the periodic voltage

$$U(0,t) \;=\; U_0 \cos(\omega t) \qquad t \geq 0$$

is fed in at the starting point $x = 0$ of a very long transmission cable. We are looking for the signal voltage $U(x,t)$ of the output signal for $x > 0, t > 0$. One obtains U as the solution of the differential equation

$$U_{tt} \;-\; c^2 U_{xx} \;+\; (\alpha+\beta)U_t \;+\; \alpha\beta U \;=\; 0\,.$$

Where α, β, c are positive parameters determined by the problem. A temporally periodic input signal leads to the expectation of a temporally periodic output signal after a certain transient phase. In addition we expect that

$$U(x,t) \quad \text{is bounded for} \quad x \to \infty\,.$$

a) Show that an approach that combines a local dampening (factor e^{-kx}) with a temporally periodic behavior (i.e. cosine/sine in t) and allows for a linear location-dependent phase shift leads to success. For example:

$$U(x,t) := e^{-kx} \cdot \Big(\delta \cos(\mu t - \gamma x) \;+\; \tilde{\delta} \sin(\tilde{\mu} t - \tilde{\gamma} x)\Big)$$

For the sake of simplicity, set $\alpha = \beta = c = 1$.

Hint: $a^2 b^2 + a^2 - b^2 - 1 = \;(a^2-1)(b^2+1)$.

b) **(Only for very fast students)**

Show that the product approach $U(x,t) = w(x) \cdot v(t)$ is not successful here.

Again, choose $\alpha = \beta = c = 1$.

Solution 2:

a) Using the ansatz: $U(x,t) := \; e^{-kx} \cdot \Big(\delta \cos(\mu t - \gamma x) \;+\; \tilde{\delta} \sin(\tilde{\mu} t - \tilde{\gamma} x)\Big)$

the boundary condition reads

$$U(0,t) = \delta \cos(\mu t) \;+\; \tilde{\delta} \sin(\tilde{\mu} t) \stackrel{!}{=} \; U_0 \cos(\omega t)\,.$$

We hence choose $\delta = U_0,\; \mu = \omega,\; \tilde{\delta} = 0$.

And obtain

$$U(x,t) = \; U_0 e^{-kx} \; \cos(\omega t - \gamma x)$$

with

$$\begin{aligned}
U_x(x,t) &= \; U_0 e^{-kx} \; [\gamma \sin(\omega t - \gamma x) - k \cos(\omega t - \gamma x)\,]\,, \\
U_t(x,t) &= \; -\omega U_0 e^{-kx} \; \sin(\omega t - \gamma x) \\
U_{xx}(x,t) &= \; U_0 e^{-kx} \; \Big[-2k\gamma \sin(\omega t - \gamma x) + (k^2 - \gamma^2) \cos(\omega t - \gamma x)\Big]\,, \\
U_{tt}(x,t) &= \; -\omega^2 U_0 e^{-kx} \; \cos(\omega t - \gamma x).
\end{aligned}$$

Inserting these terms into the differential equation , using $\alpha = \beta = c = 1$, yields

$U_0 e^{-kx} \left\{ \cos(\omega t - \gamma x) \left[-\omega^2 - (k^2 - \gamma^2) + 1 \right] + \sin(\omega t - \gamma x) \left[2k\gamma - 2\omega \right] \right\} \stackrel{!}{=} 0 \qquad \forall x, t > 0$

This kann only hold $\forall x, t > 0$ if

$k\gamma = \omega$ and $-\omega^2 - k^2 + \gamma^2 + 1 = 0$.

Inserting $k\gamma$ for ω in the second equation we obtain

$k^2\gamma^2 + k^2 - \gamma^2 - 1 = (k^2 - 1)(\gamma^2 + 1) \stackrel{!}{=} 0 \qquad$ with $\gamma \in \mathbb{R}$

$\Longrightarrow k^2 = 1, \qquad$ where we assumed $k \in \mathbb{R}^+$.

Therefore with $\quad k = 1$ and $\quad \gamma = \omega$ we have

$$U(x,t) = U_0 e^{-x} \cos(\omega(t - x)) \, .$$

b) Inserting the simple product ansatz $U(x,t) = w(x) \cdot v(t)$ into the differential equation results in

$$w(x)\ddot{v}(t) - c^2 w''(x) v(t) + (\alpha + \beta) w(x) \dot{v}(t) + \alpha\beta w(x) v(t) = 0.$$

Rearranging gives
$$\frac{\ddot{v}(t) + (\alpha+\beta)\dot{v}(t) + \alpha\beta v(t)}{v(t)} \stackrel{!}{=} c^2 \frac{w''(x)}{w(x)}$$

Since the right-hand side depends only on x and the left-hand side only on t , both sides must be constant. For example, with a fixed number $\lambda \in \mathbb{R}$:

$$\frac{\ddot{v}(t) + (\alpha+\beta)\dot{v}(t) + \alpha\beta v(t)}{v(t)} = c^2 \frac{w''(x)}{w(x)} =: -\lambda.$$

We therefore obtain a system of two ordinary differential equations coupled via the parameter λ .

The differential equation for v is

$$\ddot{v}(t) + (\alpha+\beta)\dot{v}(t) + \alpha\beta v(t) = -\lambda v(t) \iff \ddot{v} + (\alpha+\beta)\dot{v} + (\alpha\beta + \lambda) v = 0$$

This is an ordinary linear differential equation with constant coefficients. We therefore calculate the zeros of the characteristic polynomial

$$P(\mu) := \mu^2 + (\alpha+\beta)\mu + (\alpha\beta + \lambda) = 0 \iff \mu_{1,2} = -\frac{\alpha+\beta}{2} \pm \sqrt{\frac{(\alpha+\beta)^2}{4} - (\alpha\beta + \lambda)}$$

The general solution is $v(t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}$ or $v(t) = c_1 e^{\mu_1 t} + c_2 t e^{\mu_1 t}$. This is only periodic in t , if μ_1, μ_2 are purely imaginary numbers. The latter is only possible if $\alpha + \beta = 0$. But α and β are positive constants according to the task. Our product ansatz therefore does not lead to a solution.

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