Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Homework sheet 1

Exercise 1:

Consider the following differential equations for $u : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$,

$$u_t(x,t) - \epsilon u_{xx}(x,t) = 0, \qquad \epsilon \in \mathbb{R}^+, \tag{1}$$

$$u_t(x,t) + \left(\frac{(u(x,t))^2}{2}\right)_x = 0, \tag{2}$$

$$u_t(x,t) + \left(\frac{(u(x,t))^2}{2}\right)_x - \epsilon u_{xx}(x,t) = 0, \tag{3}$$

$$(u_x(x,y))^2 - (u_y(x,y))^2 - u(x,y) = 0. \tag{4}$$

a) Specify the order of each of the equations and decide whether it is a linear, semilinear, quasilinear or (fully) non-linear equation.

b) Let $u^{[1]}$ and $u^{[2]}$ be two different, non-constant solutions of the above differential equations.

For the equations (1) to (4), check whether $\tilde{u} := k \cdot u^{[1]}$ is also a solution for any $k \in \mathbb{C}$ (or $\mathbb{R}$). If yes, check whether $\hat{u} := u^{[1]} + u^{[2]}$ is a solution of the differential equation, as well.

Note that in this case, every linear combination of (any number of) solutions of the differential equation is a solution of the considered differential equation (induction argument).

Solutions to exercise 1:

a) The differential equation (1) has order 2 and is linear.

The differential equation (2) has order 1 and is quasilinear: The coefficient in front of u_x depends on u :

$$u_t(x,t) + \left(\frac{(u(x,t))^2}{2}\right)_x = u_t(x,t) + u(x,t) \cdot u_x(x,t) = 0.$$

The differential equation (3) has order 2 and is semilinear: The coefficient in front of u_{xx} does not depend on u or its derivatives.

The differential equation (4) has order 1 and is (fully) non-linear.

b) For the differential equation (1) we compute

$$u_t^{[1]} - \epsilon u_{xx}^{[1]} = 0 \Longrightarrow (k \cdot u^{[1]})_t - \epsilon (k u^{[1]})_{xx} = k[(u^{[1]})_t - \epsilon u_{xx}^{[1]}] = 0.$$

If u is a solution, then every multiple of u is also a solution.

Let $u^{[1]}$ and $u^{[2]}$ be two different solutions of differential equation (1). Then the following applies $u_t^{[1]} - \epsilon u_{xx}^{[1]} = 0$ and $u_t^{[2]} - \epsilon u_{xx}^{[2]} = 0$ and thus

$$\begin{aligned}\left(u^{[1]} + u^{[2]}\right)_t - \epsilon \left(u^{[1]} + u^{[2]}\right)_{xx} &= u_t^{[1]} + u_t^{[2]} - \epsilon \left(u_{xx}^{[1]} + u_{xx}^{[2]}\right) \\ &= \left(u_t^{[1]} - \epsilon u_{xx}^{[1]}\right) + \left(u_t^{[2]} - \epsilon u_{xx}^{[2]}\right) = 0.\end{aligned}$$

Sums of solutions of differential equation (1) also solve differential equation (1).

For a solution $u^{[1]}$ of the differential equation (2), we use $u^{[1]}_t = -\left(\frac{(u^{[1]})^2}{2}\right)_x$ for $\tilde{u} := k \cdot u^{[1]}$ and any $0 \neq k \in \mathbb{C}$ to compute

$$\begin{aligned}\tilde{u}_t + \left(\frac{(\tilde{u})^2}{2}\right)_x &= \left(k \cdot u^{[1]}\right)_t + \left(\frac{(ku^{[1]})^2}{2}\right)_x = k \cdot u^{[1]}_t + \left(\frac{k^2 \left(u^{[1]}\right)^2}{2}\right)_x \\ &= k\left(-\frac{(u^{[1]})^2}{2}\right)_x + k^2\left(\frac{\left(u^{[1]}\right)^2}{2}\right)_x = (k^2 - k)\left(\frac{(u^{[1]})^2}{2}\right)_x .\end{aligned}$$

Since $u^{[1]}$ is not constant, the differential equation only holds for $k = 1$, i.e. for $\tilde{u} = u^{[1]}$.

For the differential equation (3) and a solution $u^{[1]}$, we compute for $\tilde{u} := k \cdot u^{[1]}$ and any $0 \neq k \in \mathbb{C}$ in a completely analogous way, except that here $u^{[1]}_t = -\left(\frac{(u^{[1]})^2}{2}\right)_x + \epsilon u^{[1]}_{xx}$,

$$\begin{aligned}\tilde{u}_t + \left(\frac{(\tilde{u})^2}{2}\right)_x - \epsilon \tilde{u}_{xx} &= \left(k \cdot u^{[1]}\right)_t + \left(\frac{(ku^{[1]})^2}{2}\right)_x - \epsilon k u^{[1]}_{xx} \\ &= -k\left(\frac{(u^{[1]})^2}{2}\right)_x + k\,\epsilon u^{[1]}_{xx} + k^2\left(\frac{\left(u^{[1]}\right)^2}{2}\right)_x - \epsilon k u^{[1]}_{xx} \\ &= (k^2 - k)\left(\frac{(u^{[1]})^2}{2}\right)_x .\end{aligned}$$

Since $u^{[1]}$ is not constant, the differential equation only holds for $k = 1$, i.e. for $\tilde{u} = u^{[1]}$.

For the differential equation (4) and a solution $u^{[1]}$ we have

$$\left(u^{[1]}_x(x,y)\right)^2 - \left(u^{[1]}_y(x,y)\right)^2 = u^{[1]}(x,y)$$

and thus for $\tilde{u} := k \cdot u^{[1]}$ and any $0 \neq k \in \mathbb{C}$

$$\begin{aligned}&\left(ku^{[1]}_x(x,y)\right)^2 - \left(ku^{[1]}_y(x,y)\right)^2 - ku^{[1]}(x,y) \\ &\quad = k^2\left(u^{[1]}_x(x,y)\right)^2 - k^2\left(u^{[1]}_y(x,y)\right)^2 - ku^{[1]}(x,y) = (k^2 - k)u^{[1]}(x,y)\end{aligned}$$

Since $u^{[1]}$ is not constant, and in particular should not vanish identically, the equation only applies for $k = 1$, i.e. for $\tilde{u} = u^{[1]}$.

Exercise 2:

A simple traffic flow model:

We consider a one-dimensional flow of vehicles along an infinitely long, single-lane road. In a so-called macroscopic model, one does not consider individual vehicles, but the total flow of vehicles. For this purpose, we introduce the following quantities:

$$\begin{aligned}
u(x,t) &= \text{(length-)density of the vehicles at the point } x \text{ at the time } t \\
&= \text{vehicles/unit length at point } x \text{ at the time } t \\
v(x,t) &= \text{ speed at the point } x \text{ at the time } t \\
q(x,t) &= u(x,t)\cdot v(x,t) = \text{ flow} \\
&= \text{ amount of vehicles passing the point } x \text{ at the time } t \text{ per unit time}
\end{aligned}$$

a) Assume that there are no entrances or exits, no vehicles are disappearing, and no new vehicles are appearing. Let $N(t,a,\Delta a) :=$ number of vehicles on a spatial interval $[a, a+\Delta a]$ at time t.

   Then on the one hand it holds that

$$N(t,a,\Delta a) \;=\; \int_a^{a+\Delta a} u(x,t)\,dx$$

   and on the other hand it also holds

$$N(t,a,\Delta a) - N(t_0,a,\Delta a) \;=\; \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau)d\tau\,.$$

   Derive the so-called conservation equation for mass (number of vehicles)

$$u_t + q_x = 0$$

   from these observations.

   Hints on how to proceed:

   - Differentiate both formulas for N with respect to t. Please note that for the differentiation of parameter-dependent integrals with sufficiently smooth f the **Leibniz rule** holds:

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt \,+\, b'(x)\,f(x,b(x)) \,-\, a'(x)\,f(x,a(x))}$$

   - Divide by Δa.
   - Consider the limit $\Delta a \to 0$.

b) We now assume in a first simple model that the speed increases in inverse proportion to the density and that the density is positive

$$v(x,t) \;=\; c \,+\, \frac{k}{u(x,t)}.$$

   What is the continuity equation (=conservation equation for mass)?

Solutions to exercise 2:

a) On the one hand, it holds $\qquad N(t) \;=\; \int_a^{a+\Delta a} u(x,t)\,dx$

   and on the other hand $\qquad N(t) - N(t_0) \;=\; \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau)d\tau\,.$

   Differentiating with respect to t gives

$$\frac{\partial}{\partial t} N(t) \;=\; \frac{\partial}{\partial t}\int_a^{a+\Delta a} u(x,t)\,dx \;=\; q(a,t) - q(a+\Delta a,t).$$

   With Δa approaching zero and with sufficient smoothness of the functions, we have

$$\lim_{\Delta a \to 0} \frac{1}{\Delta a} \int_a^{a+\Delta a} \frac{\partial}{\partial t} u(x,t)\, dx = \lim_{\Delta a \to 0} -\frac{q(a+\Delta a,t) - q(a,t)}{\Delta a}$$
$$\Longrightarrow \frac{\partial}{\partial t} u(a,t) = -\frac{\partial}{\partial a} q(a,t).$$

Since these considerations hold at every point, we obtain the continuity equation

$u_t + q_x = 0\,.$

b)

$$v(x,t) \;=\; c + \frac{k}{u(x,t)} \qquad q(x,t) = \; c \cdot u(x,t) \;+\; k$$

As continuity equation we have

$$\frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0.$$

The linear transport equation is thus obtained.

Note : This is a very simple, linearized model. For example, it allows for any density and any speed. A somewhat more realistic problem would already produce shock and rarefaction waves (see later exercises).

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