Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Work sheet 6

Exercise 1:

From the lecture we know d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}\,(g(x+ct)+g(x-ct)) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\alpha)\,d\alpha$$

for solving the initial value problem for the (homogeneous) wave equation

$$\hat{u}_{tt} - c^2\hat{u}_{xx} = 0,\ \hat{u}(x,0) = g(x),\ \hat{u}_t(x,0) = h(x),\ x \in \mathbb{R},\ c > 0.$$

The function

$$\tilde{u}(x,t) = \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} f(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau \qquad (1)$$

solves the following initial value problem

$$\tilde{u}_{tt} - c^2\tilde{u}_{xx} = f(x,t) \qquad \tilde{u}(x,0) = \tilde{u}_t(x,0) = 0\,. \qquad (2)$$

(Proof: Leibniz formula for the derivation of parameter-dependent integrals)

Now we consider the initial value problem

$$\begin{aligned} &u_{tt} - 4u_{xx} = 6x\sin t, \qquad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = x,\ x \in \mathbb{R}, \qquad u_t(x,0) = \sin(x),\ x \in \mathbb{R} \end{aligned} \qquad (3)$$

a) Compute a solution $\hat{u}$ to the IVP

$$\begin{aligned} &\hat{u}_{tt} - 4\hat{u}_{xx} = 0, && x \in \mathbb{R},\ t > 0 \\ &\hat{u}(x,0) = x,\ x \in \mathbb{R}, && \hat{u}_t(x,0) = \sin(x),\ x \in \mathbb{R}. \end{aligned}$$

b) Compute a solution $\tilde{u}$ to the IVP

$$\begin{aligned} &\tilde{u}_{tt} - 4\tilde{u}_{xx} = 6x\sin t, && x \in \mathbb{R},\ t > 0 \\ &\tilde{u}(x,0) = 0,\ x \in \mathbb{R}, && \tilde{u}_t(x,0) = 0,\ x \in \mathbb{R} \end{aligned}$$

c) By inserting u into the differential equation and checking the initial values, show that $u = \tilde{u} + \hat{u}$ solves the initial value problem (3).

Exercise 2: (Vibrating String)

Solve the initial boundary value problem

$$\begin{aligned}
u_{tt} &= c^2 u_{xx} && \text{for} \quad 0 < x < 1,\ t > 0,\\
u(0,t) &= u(1,t) \ = \ 0 && \text{for} \quad t > 0,\\
u(x,0) &= 0 && \text{for} \quad 0 < x < 1,\\
u_t(x,0) &= \begin{cases} 1, & \frac{1}{20} \le x \le \frac{1}{10},\\ 0 & \text{else.} \end{cases}
\end{aligned}$$

Hint: Lecture page 150.

You will receive a series as a solution. Plot the partial sums of the first 20 non-vanishing summands of this series for $c = 2$, $x \in [0,1], t \in [0, 0.4]$ and $t \in [0, 2]$.

Bearbeitung: 07.07.- 10.07.2025