Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
Prof. Dr. J. Struckmeier
Dr. H. P. Kiani

Differential Equations II for Engineering Students

Work sheet 4

Exercise 1: See Lecture pages 47-53

Consider the initial value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} = \ 0 \quad &\text{for } x \in \mathbb{R},\ t \in \mathbb{R}^+ \\ u(x,0) = 0 \quad &\text{for } x \in \mathbb{R}, \\ u_t(x,0) = 2xe^{-x^2} \quad &\text{for } x \in \mathbb{R}. \end{aligned}$$

a) Rewrite the PDE in matrix form.

b) Carry out the substitution $\alpha = x + \frac{t}{4}$, $\mu = x - t$ and give the PDE in matrix notation for $v(\alpha\,,\mu) := u(x,t)$.

c) Solve the PDE for u by first solving the PDE for $\boldsymbol{v}$ and transforming back afterwards.

d) Determine the solution u for the initial value problem.

Exercise 2: Hint: See lecture page 60 and 65.

a) Let α be a fixed real number from $\mathbb{R}\setminus\{0\}$. For which real-valued functions $g : \mathbb{R} \to \mathbb{R}$ are the following functions harmonic in $\mathbb{R}^2$?

   i) $\tilde{u}(x,y) = \cos(\alpha x) \cdot g(y)$, ii) $u(x,y) = \frac{1}{2} \cdot (x^3 + g(x) \cdot y^2)$.

b) Let $\Omega := \left\{(x,y)^T \in \mathbb{R}^2 : x^2 + y^2 < 16\right\}$ and u be the solution of the boundary value problem

$$\Delta u(x,y) = 0 \qquad \text{in } \Omega, \qquad u(x,y) = \frac{2y^2}{x^2+y^2} \qquad \text{on} \quad \partial\Omega.$$

Determine the value of u in the origin.

- Use polar coordinates and the mean value property (lecture page 65).
- Note: $\sin^2(\varphi) = \dfrac{1-\cos(2\varphi)}{2}$.

Exercise 3: Hint: Lecture pages 61-64 and 69

a) Let

$$\Omega_2 = \{(x,y)^\top \in \mathbb{R}^2 \ : \ 1 < x^2 + y^2 < 4\}.$$

Determine the solutions of

$$\begin{cases} \Delta u = 0 & \text{on} & \Omega_2, \\ u(x,y) = 1 & \text{for} & x^2 + y^2 = 1, \\ u(x,y) = 3 & \text{for} & x^2 + y^2 = 4. \end{cases}$$

Is the solution unique?

b) Let

$$\Omega_3 = \{(x,y,z)^\top \in \mathbb{R}^3 \ : \ 1 < x^2 + y^2 + z^2 < 4\}.$$

Determine the solutions of

$$\begin{cases} \Delta u = 0 & \text{on} & \Omega_3, \\ u(x,y,z) = 1 & \text{for} & x^2 + y^2 + z^2 = 1, \\ u(x,y,z) = 3 & \text{for} & x^2 + y^2 + z^2 = 4. \end{cases}$$

Is the solution unique?

Discussion: 09.06.- 13.06.2025