Fachbereich Mathematik der Universität Hamburg **SuSe 2025**
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Differential Equations II for Engineering Students

Homework sheet 3

Exercise 1:

a) Consider the following initial-value problem for $u(x,t)$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$

$$\begin{aligned} u_t + u \cdot u_x &= 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+ \\ u(x,0) &= g(x), \qquad x \in \mathbb{R}\,. \end{aligned}$$

Here let $g : \mathbb{R} \to \mathbb{R}$ be a strictly monotonically increasing function with two points of discontinuity (jump points).

For each of the following statements, determine if it is true or false.

(i) There is a unique weak solution.

(ii) In order to obtain the entropy solution, one has to introduce two shock waves.

(iii) The entropy solution is valid for all times, i.e. for any $t \in \mathbb{R}^+$.

Justify your answers.

b) What is the jump condition for the weak solution to

$$\begin{aligned} u_t + (u^3)_x &= 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+ \\ u(x,0) &= \begin{cases} 4 & \text{for } x \leq 0, \\ 2 & \text{for } x > 0\,? \end{cases} \end{aligned}$$

Exercise 2: Determine the entropy solution to the Burgers' equation $u_t + uu_x = 0$ with the initial data

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

at the time $t = 2$. What new problem occurs at $t = 2$?

Additional voluntary task: Determine the solution for $t > 2$.

Exercise 3:

We discuss again the simple traffic flow model from Sheet 1 with the notation introduced there:

$u(x,t)\ =\ $ density of vehicles (vehicles/length) at point x at time t,

$v(x,t)\ =\ $ velocity at point x at time t,

$q(x,t)\ =\ u(x,t)\cdot v(x,t) =$ flow = number of vehicles passing x at time t per time unit.

We improve our model from Sheet 1 by incorporating maximal density and a maximal velocity

$u_{max}\ =\ $ maximal density of vehicles (bumper to bumper),

$v_{max}\ =\ $ maximal velocity

This can be done, for example, as follows:

$$v(x,t) := v(u(x,t)) := v_{max}\left(1 - \frac{u(x,t)}{u_{max}}\right)$$

a) Set up the continuity equation ($u_t + q_x = 0$).

b) Show again that the characteristics are straight lines and determine their slopes.

c) Sketch the characteristics for

$$v_{max} = 1 \quad \text{(Here it has been scaled appropriately!)}$$

$$u(x,0) = \begin{cases} u_l = u_{max}/2 & x < 0 \\ u_r = u_{max} & x > 0 \quad \text{(red traffic light/traffic jam etc.)} \end{cases}$$

d) For the Burgers' equation we allowed shock waves only in the case $u_l > u_r$. There must obviously be a different condition here. What could be the reason for that?

   Note: This question can not be answered completely only with help of the lecture slides. You can only make a guess here!

Submission deadline: 23.05.2025