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Auditorium Exercise 06

Differential Equations II for Students of Engineering Sciences
Summer Semester 2025

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July 4 2025

Universität Hamburg
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Overview

- Wave Equation
  - Homogeneous initial value problem
  - Inhomogeneous initial value problem
  - Homogeneous initial boundary value problem
  - Inhomogeneous initial boundary value problem

Wave equation

Homogeneous initial value problem (IVP) in $\mathbb{R}$ (Cauchy problem)

$$u_{tt} - c^2 u_{xx} = 0 \qquad x \in \mathbb{R}, \quad t > 0 \quad c > 0,$$
$$u(x,0) = u_0(x) = g(x), \quad u_t(x,0) = v_0(x) = h(x) \qquad x \in \mathbb{R}.$$

d'Alembert's formula

$$\boxed{u(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\psi)\, d\psi\,.}$$

Derivation: By substitution of (compare Exercise 1, Sheet 4)

$$\alpha = x + ct,\ \mu = x - ct,$$

and

$$w(\alpha(x,t), \mu(x,t)) = u(x,t)$$

the chain rule yields $u_{tt} - c^2 u_{xx} \iff w_{\alpha\mu} = 0$ (Integrable form)

Homogeneous IVP

Precisely: $\left(w_\alpha(\alpha,\mu)\right)_\mu = 0$.

$$w_\alpha(\alpha,\mu) = \phi(\alpha) \implies w(\alpha,\mu) = \Phi(\alpha) + \Psi(\mu)$$

$$\implies u(x,t) = \Phi(x+ct) + \Psi(x-ct)$$

Initial conditions:

$$u(x,0) = \Phi(x) + \Psi(x) = g(x), \qquad u_t(x,0) = c\Phi'(x) - c\Psi'(x) = h(x)$$

Deriving the first equation yields

$$c\Phi'(x) + c\Psi'(x) = cg'(x), \qquad u_t(x,0) = c\Phi'(x) - c\Psi'(x) = h(x)$$

These are two equations for Φ' and Ψ'. Solving yields d'Alembert's formula.

Homogeneous IVP

Proof left to the reader

Add the last two equations

$$2c\Phi'(x) = cg'(x) + h(x) \implies \Phi'(x) = \frac{1}{2}g'(x) + \frac{1}{2c}h(x)$$

$$\implies \Phi(x) = \frac{1}{2}g(x) + B + \frac{1}{2c}\int_{x_0}^{x} h(\sigma)d\sigma\,,$$

$$\Psi(x) = g(x) - \Phi(x) = \frac{1}{2}g(x) - B - \frac{1}{2c}\int_{x_0}^{x} h(\sigma)d\sigma$$

$$\implies \Phi(x+ct) = \frac{1}{2}g(x+ct) + B + \frac{1}{2c}\int_{x_0}^{x+ct} h(\sigma)d\sigma$$

$$\Psi(x-ct) = \frac{1}{2}g(x-ct) - B - \frac{1}{2c}\int_{x_0}^{x-ct} h(\sigma)d\sigma$$

Homogeneous IVP

Proof continued

$$
\begin{aligned}
u(x,t) &= \Phi(x+ct) + \Psi(x-ct) \\
&= \frac{1}{2}g(x+ct) + \frac{1}{2c}\int_{x_0}^{x+ct} h(\sigma)d\sigma + \frac{1}{2}g(x-ct) - \frac{1}{2c}\int_{x_0}^{x-ct} h(\sigma)d\sigma \\
&= \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\left(\int_{x_0}^{x+ct} h(\sigma)d\sigma + \int_{x-ct}^{x_0} h(\sigma)d\sigma\right) \\
&= \frac{1}{2}\left(g(x+ct) + g(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\sigma)d\sigma\,.
\end{aligned}
$$

Wave equation

Inhomogeneous initial value problem (IVP) in $\mathbb{R}$ (Cauchy problem)

$$\begin{aligned} \tilde{u}_{tt} - c^2 \tilde{u}_{xx} &= h(x,t) \qquad x \in \mathbb{R}, \quad t > 0 \quad c > 0, \\ \tilde{u}(x,0) &= \tilde{u}_t(x,0) = 0 \qquad x \in \mathbb{R}. \end{aligned}$$

Solution:

$$\tilde{u}(x,t) = \frac{1}{2c} \int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\, \mathrm{d}\omega \mathrm{d}\tau \tag{1}$$

Inhomogeneous IVP

Proof left to the reader

By Leibniz formula for parameter-dependent integrals

$$\boxed{\frac{d}{dy} \int_{a(y)}^{b(y)} f(y,z)dz = \int_{a(y)}^{b(y)} \frac{d}{dy} f(y,z)dz + b'(y)f(y,b(y)) - a'(y)f(y,a(y))}$$

one computes

$$\begin{aligned}
\tilde{u}_x(x,t) &= \frac{d}{dx}\left(\frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\right) d\omega d\tau \\
&= \frac{1}{2c}\int_0^t \frac{d}{dx}\left(\int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\, d\omega\right) d\tau \\
&= \frac{1}{2c}\int_0^t \left[h(x-c(\tau-t),\tau) - h(x+c(\tau-t),\tau\right] d\tau
\end{aligned}$$

$$\tilde{u}_{xx}(x,t) = \frac{1}{2c}\int_0^t \left[h_\omega(x-c(\tau-t),\tau) - h_\omega(x+c(\tau-t),\tau\right] d\tau$$

Inhomogeneous IVP

Proof continued

$$\begin{aligned}
\tilde{u}_t(x,t) &= \frac{d}{dt}\left(\frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\right) \mathrm{d}\omega \mathrm{d}\tau \\
&= \frac{1}{2c}\int_0^t \frac{d}{dt}\left(\int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\,\mathrm{d}\omega\,\mathrm{d}\tau\right) \\
&+ \frac{1}{2c}(\dot{t})\int_{x+c(t-t)}^{x-c(t-t)} h(\omega,t)\,\mathrm{d}\omega \\
&= \frac{1}{2c}\int_0^t \left[h(x-c(\tau-t),\tau)\cdot c - h(x+c(\tau-t),\tau)\cdot(-c)\right] \mathrm{d}\tau \\
&= \frac{1}{2}\int_0^t \left[h(x-c(\tau-t),\tau) + h(x+c(\tau-t),\tau)\right] \mathrm{d}\tau
\end{aligned}$$

Inhomogeneous IVP

Proof continued

$$\begin{aligned}
\tilde{u}_{tt}(x,t) &= \frac{1}{2}\int_0^t \frac{d}{dt}\left[h(x-c(\tau-t),\tau)+h(x+c(\tau-t),\tau)\right]d\tau \\
&+ \frac{1}{2c}(\dot{t})\left[h(x-c(t-t),\tau)+h(x+c(t-t),\tau)\right] \\
&= \frac{1}{2}\,\{h(x,t)+h(x,t) \\
&\quad + \int_0^t \left[h_\omega(x-c(\tau-t),\tau)\cdot c + h_\omega(x+c(\tau-t),\tau)(-c)\right]\,d\tau\} \\
&\quad = h(x,t) + \frac{c}{2}\int_0^t \left[h_\omega(x-c(\tau-t),\tau) - h_\omega(x+c(\tau-t),\tau)\right]\,d\tau
\end{aligned}$$

Obviously it holds that $\tilde{u}_{tt} - c^2\tilde{u}_{xx} = h(x,t)$. *For the initial values one gets*

$$\tilde{u}(x,0) = \frac{1}{2c}\int_0^0 \cdots = 0 \qquad \textit{and} \qquad \tilde{u}_t(x,0) = \frac{1}{2}\int_0^0 \cdots = 0.$$

Inhomogeneous IVP

Example

$$u_{tt} - 9u_{xx} = -4x \qquad x \in \mathbb{R}, \quad t > 0\,,$$
$$u(x,0) = 1 \qquad u_t(x,0) = \cos(x) \qquad x \in \mathbb{R}.$$

Method: Solve two equations and combine. Precisely:

- Inhomogeneous DE with homogeneous initial values:

$$\tilde{u}_{tt} - 9\tilde{u}_{xx} = -4x \qquad x \in \mathbb{R}, \quad t > 0\,,$$
$$\tilde{u}(x,0) = 0 \qquad \tilde{u}_t(x,0) = 0 \qquad x \in \mathbb{R}.$$

- Homogeneous DE with inhomogeneous initial values:

$$\hat{u}_{tt} - 9\hat{u}_{xx} = 0 \qquad x \in \mathbb{R}, \quad t > 0\,,$$
$$\hat{u}(x,0) = 1 \qquad \hat{u}_t(x,0) = \cos(x) \qquad x \in \mathbb{R}.$$

- Solution of original problem: $u = \hat{u} + \tilde{u}$.

Solution:

- $\tilde{u}_{tt} - 9\tilde{u}_{xx} = -4x$

  $\tilde{u}(x,t) = \dfrac{1}{2c} \int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau$

  where $h(x,t) = -4x$ hence $h(\omega,\tau) =$ and $c =$

  Hence

  $\tilde{u}(x,t) =$

Inhomogeneous IVP
Example

- $\hat{u}_{tt} - 9\hat{u}_{xx} = 0, \qquad \hat{u}(x,0) = 1, \qquad \hat{u}_t(x,0) = \cos(x)\,.$

  $$\hat{u}(x,t) = \frac{1}{2}\left[g(x+ct) + g(x-ct)\right] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\psi)\,d\psi$$

  $g(x) = u_0(x) = \qquad\qquad h(x) = v_0(x) =$

  $\hat{u}(x,t) =$

- Claim: $u = \tilde{u} + \hat{u}$ solves the original problem:

  $$u_{tt} - 9u_{xx} = -4x \qquad x \in \mathbb{R}, \quad t > 0\,,$$
  $$u(x,0) = 1 \qquad u_t(x,0) = \cos(x) \qquad x \in \mathbb{R}.$$

Verification:

$$u(x,t) = -2xt^2 + 1 + \frac{1}{6}\left[\sin(x+3t) - \sin(x-3t)\right]$$

$u(x,0) =$

$u_t(x,t) =$

$u_t(x,0) =$

$u_{xx} =$

$u_{tt} =$

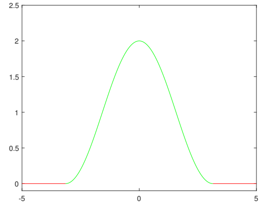
$u_{tt} - 9u_{xx} =$

Example for Homework 1b

$$u_{tt} = 4u_{xx}, \quad \text{for } x \in \mathbb{R},\ t > 0\,,$$

$$u(x,0) = u_0(x) = g(x) = \begin{cases} 1 + \cos(x) & -\pi \le x \le \pi\,, \\ 0 & \text{else,} \end{cases}$$

$$u_t(x,0) = 0.$$



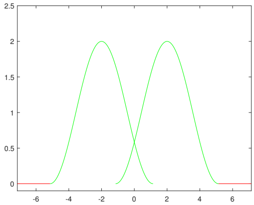
$$u(x,t) =$$

$$g(x+2t) =$$

Example for Homework 1b
continued

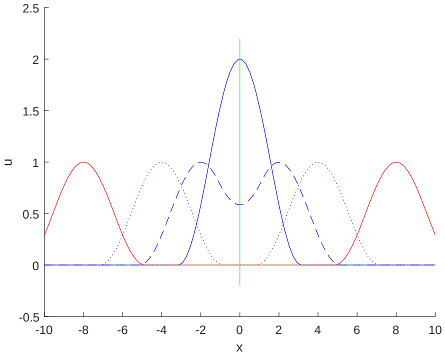
$$g(x-2t) = \begin{cases} 1+\cos(x-2t) & -\pi \leq x-2t \leq \pi \iff x \in [-\pi+2t, \pi+2t] \\ 0 & \text{else,} \end{cases}$$

For example for $t = 1$:



Example for Homework 1b
continued

for $t = 0, 1, 2, 4,$



1: dashed, 2: dotted, 4: red

Homogeneous Wave Equation with homogeneous BV

Initial boundary value problem

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 & & c > 0,\ t > 0,\ x \in (0, L) \\
u(x,0) &= u_0(x) & & x \in (0, L), \\
u_t(x,0) &= v_0(x) & & x \in (0, L), \\
u(0,t) &= 0 & & t > 0, \\
u(L,t) &= 0 & & t > 0
\end{aligned}
$$

Product ansatz

- Ansatz $u(x,t) = q(t) \cdot p(x)$
  yields $p(x) \cdot \ddot{q}(t) = c^2 p''(x) \cdot q(t)$
- $\Longrightarrow \dfrac{p''}{p} = \dfrac{\ddot{q}}{c^2 q} = -\lambda$
- $p'' = -\lambda p$ and $\ddot{q} = -\lambda c^2 q$

Homogeneous Wave Equation with homogeneous BVs
$p'' = -\lambda p$

By the homogeneous boundary values we get, exactly as for the heat equation,

$$u(0,t) = p(0)q(t) = 0 \quad \forall t > 0 \quad \Longrightarrow \quad p(0) = 0 \vee q \equiv 0$$

$$u(L,t) = p(L)q(t) = 0 \quad \forall t > 0 \quad \Longrightarrow \quad p(L) = 0 \vee q \equiv 0.$$

the boundary value problem:

$$p''(x) = -\lambda p(x), \qquad p(0) = p(L) = 0$$

The only solutions that are nontrivial are (compare AE 5).

$$\boxed{p_k(x) = \sin(k\omega x) \qquad \omega = \pi/L, \quad \lambda_k = \left(\frac{k\pi}{L}\right)^2 = (k\omega)^2, \quad k \in \mathbb{N}}$$

Homogeneous Wave Equation with homogeneous BVs

$q = -\lambda c^2 q$

For q we have the following DE this time:

$$\ddot{q} = -\lambda c^2 q = -(ck\omega)^2 q$$

which gives

$$q_k(t) = A_k \cos(ck\omega t) + B_k \sin(ck\omega t)$$

Hence $\quad u_k(x,t) := q_k(t) \cdot p_k(x), \qquad k \in \mathbb{N}$ solves the DE and fulfills the boundary values.

DE is homogeneous and linear, homogeneous bvs $\longrightarrow$ superposition allowed

$$\boxed{u(x,t) = \sum_{k=1}^{n} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x)}$$

solves the DE and fulfills the boundary values.

Homogeneous Wave Equation with homogeneous BVs

Initial conditions

With $u(x,t) = \sum_{k=1}^{n} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x)$
the initial values

$$u(x,0) = \sum_{k=1}^{n} \left(A_k \cos(0) + B_k \sin(0)\right) \cdot \sin(k\omega x) = u_0(x) \qquad x \in [0, L]$$

have to be fulfilled.

For $n \to \infty$ we get

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin(k\omega x) = u_0(x), \qquad x \in [0, L].$$

A_k are the Fourier coefficients of the odd $2L$-periodic continuation of u_0
if the initial data is smooth

$$A_k = \frac{2}{L} \int_0^L u_0(\alpha) \sin\left(\frac{k\pi\alpha}{L}\right) d\alpha$$

The second initial condition is for

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x)$$

$$u_t(x,t) =$$

$$u_t(x,0) = \sum_{k=1}^{\infty} B_k \cdot (ck\omega) \sin(k\omega x) \stackrel{!}{=} v_0(x).$$

With the Fourier coefficients of the odd $2L$-periodic continuation of v_0

$$b_k = \frac{2}{L} \int_0^L v_0(\alpha) \sin\left(\frac{k\pi\alpha}{L}\right) d\alpha$$

it needs to hold that

$$\sum_{k=1}^{\infty} B_k \cdot \frac{ck\pi}{L} \sin\left(\frac{k\pi}{L}x\right) = \sum_{k=1}^{\infty} b_k \sin\left(\frac{k\pi}{L}x\right)$$

Homogeneous Wave Equation with homogeneous BVs

Continued

Therewith we get the solution

$$u(x,t) = \sum_{k=1}^{\infty} \left(A_k \cos(ck\omega t) + B_k \sin(ck\omega t)\right) \cdot \sin(k\omega x) \qquad \omega = \tfrac{\pi}{L}$$

$$A_k = \frac{2}{L} \int_0^L u_0(\alpha) \sin\left(\frac{k\pi\alpha}{L}\right) d\alpha, \qquad B_k = \frac{2}{ck\pi} \int_0^L v_0(\alpha) \sin\left(\frac{k\pi\alpha}{L}\right) d\alpha$$

Homogeneous Wave Equation with homogeneous BVs

Example

$$u_{tt} - u_{xx} = 0 \qquad x \in (0, \frac{\pi}{2}),\ t > 0$$

$$u(x,0) = \begin{cases} x & x \in [0, \frac{\pi}{4}], \\ \frac{\pi}{2} - x & x \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases}$$

$$u_t(x,0) = 2\sin(4x) \qquad x \in [0, \frac{\pi}{2}],$$

$$u(0,t) = u(\frac{\pi}{2}, t) = 0 \qquad t > 0,$$

Here it holds: $L = \frac{\pi}{2}, \ \ \omega = \frac{\pi}{L} = \frac{\pi}{\pi/2} = 2, \ \ c = 1$

$$u(x,t) = \sum_{k=1}^{\infty} [A_k \cos(ck\omega t) + B_k \sin((ck\omega t)] \sin(k\omega x),$$

As above:

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin(k\omega x) \stackrel{!}{=} \begin{cases} x & x \in [0, \frac{\pi}{4}], \\ \frac{\pi}{2} - x & x \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases}$$

Homogeneous Wave Equation with homogeneous BVs

Example continued

The second initial condition yields

$$u_t(x,0) = \sum_{k=1}^{\infty} B_k \cdot (ck\omega)\sin(k\omega x) = 2\sin(4x)$$

with $\omega = 2, \;\; c = 1.$

Obviously:

The first initial condition requires

$$u(x,0) = \sum_{k=1}^{\infty} A_k \cdot \sin(2kx) = \begin{cases} x & x \in [0, \frac{\pi}{4}], \\ \frac{\pi}{2} - x & x \in [\frac{\pi}{4}, \frac{\pi}{2}] \end{cases}$$

Homogeneous Wave Equation with homogeneous BVs

Example continued

For A_k we compute

$$
\begin{aligned}
A_k &= \frac{2}{L} \int_0^L u(x,0) \cdot \sin(2kx)\, dx \\
&= \frac{4}{\pi} \int_0^{\frac{\pi}{4}} x \sin(2kx) dx + \frac{4}{\pi} \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (\frac{\pi}{2} - x) \sin(2kx) dx \\
&= \frac{4}{\pi} \Big\{ \Big[x(-\frac{\cos(2kx)}{2k}) \Big]_0^{\frac{\pi}{4}} - \int_0^{\frac{\pi}{4}} -\frac{\cos(2kx)}{2k} dx \\
&+ \Big[(\frac{\pi}{2} - x)(-\frac{\cos(2kx)}{2k}) \Big]_{\frac{\pi}{4}}^{\frac{\pi}{2}} - \int_{\frac{\pi}{4}}^{\frac{\pi}{2}} (-1) \left(-\frac{\cos(2kx)}{2k} \right) dx \Big\} \\
&\vdots \\
&= \frac{2}{k^2 \pi} \sin\left(\frac{k\pi}{2} \right)
\end{aligned}
$$

Homogeneous Wave Equation with homogeneous BVs

Example continued

Hence we get

$$
\begin{aligned}
u(x,t) &= \sum_{k=1}^{\infty} [A_k \cos(ck\omega t) + B_k \sin((ck\omega t)] \sin(k\omega x) \\
&= \left[\sum_{k=1}^{\infty} \frac{2}{k^2\pi} \sin\left(\frac{k\pi}{2}\right) \cos(ck\omega t) \sin(k\omega x) \right] + B_2 \sin(2c\omega t) \sin(2\omega x) \\
&= \left[\sum_{k=1}^{\infty} \frac{2}{k^2\pi} \sin\left(\frac{k\pi}{2}\right) \cos(2kt) \sin(2kx) \right] + \frac{1}{2} \sin(4t) \sin(4x) \\
&= \frac{1}{2} \sin(4t) \sin(4x) + \sum_{k=1}^{\infty} \frac{2}{k^2\pi} \sin\left(\frac{k\pi}{2}\right) \cos(2kt) \sin(2kx) \cdot
\end{aligned}
$$

Homogeneous Wave Equation with inhomogeneous BVs

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= 0 && c > 0,\ x \in (0, L),\ t > 0 \\
u(x, 0) &= u_0(x) && x \in [0, L], \\
u_t(x, 0) &= w_0(x) && x \in [0, L], \\
u(0, t) &= h(t) && t \geq 0, \\
u(L, t) &= g(t) && t \geq 0,
\end{aligned}
$$

Proceed as in auditorium exercise 5:

$$v(x, t) := u(x, t) - h(t) - \frac{x}{L}\left(g(t) - h(t)\right)$$

yields $v(0, t) = v(L, t) = 0$.

New DE for v:

$$u(x, t) := v(x, t) + h(t) + \frac{x}{L}\left(g(t) - h(t)\right)$$

$$u_{tt}(x, t) =$$

$$u_{xx}(x, t) := v_{xx}(x, t)$$

Homogeneous Wave Equation with inhomogeneous BVs

New problem

The new problem consists of: in general, inhomogeneous DE, inhomogeneous initial values but **homogeneous boundary values**

$$
\begin{aligned}
& v_{tt} + \ddot{h}(t) + \frac{x}{L}\left(\ddot{g}(t) - \ddot{h}(t)\right) - c^2 v_{xx} = 0 \\
& v(x,0) = u(x,0) - h(0) - \frac{x}{L}\left(g(0) - h(0)\right) =: v_0(x). \\
& v_t(x,0) = u_t(x,0) - \dot{h}(0) - \frac{x}{L}\left(\dot{g}(0) - \dot{h}(0)\right) =: \hat{v}_0(x). \\
& v(0,t) = 0\,, \qquad v(L,t) = 0\,.
\end{aligned}
$$

Inhomogeneous Wave Equation with inhomogeneous BVs

Example

$$
\begin{aligned}
u_{tt} &= 4u_{xx} && c > 0,\ t > 0,\ x \in (0, L) \\
u(x,0) &= x - \sin(\pi x) && x \in (0, L), \\
u_t(x,0) &= \sin(2\pi x) && x \in (0, L), \\
u(0,t) &= 0 && t > 0, \\
u(L,t) &= 1 && t > 0
\end{aligned}
$$

Transform the equation into an initial boundary value problem with homogeneous boundary values by introducing an appropriate function v.

Solve the the IBVP for v

Give the solution of the IBVP for u.

1. Homogenization

   $$v(x,t) := u(x,t) - h(t) - \frac{x}{L}\left(g(t) - h(t)\right)$$

   $$= u(x,t) - 0 - \frac{x}{1}(1-0) = u(x,t) - x \cdot$$

   $v_{xx} =$ $\qquad$ $v_{tt} =$

Inhomogeneous Wave Equation with inhomogeneous BVs

Example

2. New Problem

$$
\begin{aligned}
v_{tt} &= 4v_{xx} && 0 < x < 1,\ t \in \mathbb{R}^+, \\
v(x,0) &= u(x,0) - x = -\sin(\pi x) && 0 \le x \le 1\,, \\
v_t(x,0) &= u_t(x,0) = \sin(2\pi x) && 0 \le x \le 1\,, \\
v(0,t) &= u(0,t) - 0 = 0 && t \ge 0\,, \\
v(1,t) &= u(1,t) - 1 = 0 && t \ge 0\,.
\end{aligned}
$$

3. Solution of IBVP for v:
   Use $c = 2, \quad L = 1$

$$
v(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{ck\pi}{L}\, t\right) + B_k \sin\left(\frac{ck\pi}{L}\, t\right) \right] \sin\left(\frac{k\pi}{L}\, x\right)
$$

3. continued

$v(x, 0) =$

$$v_t(x,t) = \sum_{k=1}^{\infty} \left[-2k\pi A_k \sin(2k\pi\, t) \,+\, 2k\pi B_k \cos(2k\pi\, t) \right] \sin(k\pi x)$$

$v_t(x, 0) =$

and therefore

$v(x,t) \;=\; A_1 \cos\left(2 \cdot 1 \cdot \pi \cdot t\right) \sin\left(1 \cdot \pi\, x\right) + B_2 \sin\left(2 \cdot 2 \cdot \pi \cdot t\right) \sin\left(2 \cdot \pi \cdot x\right)$

hence

$$v(x,t) \;=\; \frac{1}{4\pi} \sin(4\pi t) \sin(2\pi x) - \cos(2\pi t) \sin(\pi x)$$

Inhomogeneous Wave Equation with inhomogeneous BVs

Example

4. Solution of IBVP for u:

   The solution of the original problem is

$$u(x,t) = v(x,t) + x = x + \frac{1}{4\pi}\sin(4\pi t)\sin(2\pi x) - \cos(2\pi t)\sin(\pi x)$$

Inhomogeneous Wave Equation with homogeneous BVs

for completeness, left to the reader

$$u_{tt} - c^2 u_{xx} = h(x,t) \qquad c > 0,\ x \in (0,L),\ t > 0$$

$$u(x,0) = u_0(x) \qquad x \in (0,L),$$

$$u_t(x,0) = v_0(x) \qquad x \in (0,L),$$

$$u(0,t) = 0 \qquad t > 0,$$

$$u(L,t) = 0 \qquad t > 0,$$

Let $\omega = \frac{\pi}{L}$

$$\boxed{u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin(k\omega x)}$$

Inhomogeneous Wave Equation with homogeneous BVs

Plugging this into the PDE gives boundary value problems for ordinary differential equations of second order, if the series converge uniformely:

$$\ddot{q}_k(t) + c^2 k^2 \omega^2 q_k(t) \;=\; c_k(t)\,, \qquad q_k(0) = a_k,\; q_k'(0) = b_k$$

where:

$$a_k = \frac{2}{L}\int_0^L u_0(x)\sin(k\omega x)\,dx$$
$$b_k = \frac{2}{L}\int_0^L v_0(x)\sin(k\omega x)\,dx$$
$$c_k(t) = \frac{2}{L}\int_0^L h(x,t)\sin(k\omega x)\,dx$$

Fourier coefficients might be computed by comparing coefficients!