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Auditorium Exercise 05

Differential Equations II for Students of Engineering Sciences
Summer Semester 2025

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Overview

- Heat Equation
  - Homogeneous Heat Equation
  - Inhomogeneous Heat Equation
- Laplace Equation
  - Rotationally Symmetric Data
  - Not Rotationally Symmetric Data

Initial Boundary Value Problem for the Heat Equation

We want to solve

$$\begin{aligned} u_t - cu_{xx} &= h(x,t), && c > 0,\ t > 0,\ x \in (a,b),\ \text{here } (0,L),\\ u(x,0) &= u_0(x), && x \in (a,b),\\ u(a,t) &= f(t), && t > 0,\\ u(b,t) &= g(t), && t > 0. \end{aligned}$$

c: Diffusion coefficient/ heat conduction

Homogeneous Heat Equation
with homogeneous BV and inhomogeneous IV

$$\begin{aligned}
\tilde{v}_t - c\tilde{v}_{xx} &= 0, && c > 0,\ t > 0,\ x \in (0, L),\ L > 0,\\
\tilde{v}(x, 0) &= v_0(x), && x \in (0, L),\\
\tilde{v}(0, t) &= \tilde{v}(L, t) = 0, && t > 0.
\end{aligned}$$

v_0 is nonzero, that is, $\tilde{v}$ is nonzero!

- Ansatz: $\tilde{v}(x, t) = q(t) \cdot p(x)$
- Insertion into PDE gives $\quad \dot{q}(t) \cdot p(x) - c \cdot q(t) \cdot p''(x) = 0$
- Reorder:

$$\frac{\dot{q}(t)}{q(t)} = c\,\frac{p''(x)}{p(x)} =$$

Homogeneous Heat Equation

First: $p''(x) = -\lambda \cdot p(x)$ (compare lecture notes pages 85-88)

$\tilde{v}(0,t) = q(t) \cdot p(0) \stackrel{!}{=} 0 \implies p(0) =$

$\tilde{v}(L,t) = q(t) \cdot p(L) \stackrel{!}{=} 0 \implies p(L) =$

DE: $p'' + \lambda \cdot p = 0 \longrightarrow$ Characteristic polynomial: $\mu^2 + \lambda = 0$

$\mu = \pm\sqrt{-\lambda} \longrightarrow$ general solution $ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$

Except for double roots! Here $\lambda = 0$

$$\lambda = 0 \Longrightarrow p(x) = a_0 e^{\sqrt{-0}x} + b_0 x e^{\sqrt{-0}x} = a_0 + b_0 x,$$
$$p(0) = 0 \Longrightarrow a_0 = 0$$
$$p(L) = 0 \Longrightarrow b_0 \cdot L = 0$$

Homogeneous Heat Equation

$$\lambda < 0 \implies p(x) = ae^{\sqrt{-\lambda}x} + be^{-\sqrt{-\lambda}x}$$

$$p(0) = 0 \implies ae^0 + be^0 = 0$$

$$p(L) = 0 \implies ae^{\sqrt{-\lambda}L} + be^{-\sqrt{-\lambda}L} = 0$$

$$\lambda > 0 \implies p(x) = \hat{a}e^{\sqrt{-\lambda}x} + \hat{b}e^{-\sqrt{-\lambda}x}$$
$$p(x) = \hat{a}e^{i\sqrt{\lambda}x} + \hat{b}e^{-i\sqrt{\lambda}x}$$

Real representation: $p(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x)$

$$p(0) = 0 \implies a\cos(0) + b\sin(0) = a = 0$$

$$p(L) = 0 \implies b\sin(\sqrt{\lambda}\,L) = 0$$

So nontrivial solutions exist only for:

$$\boxed{\lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2\omega^2, \qquad n \in \mathbb{N},\ \omega = \frac{\pi}{L}}$$

Homogeneous Heat Equation

$$\lambda_n = \left(\frac{n\pi}{L}\right)^2 = n^2\omega^2, \qquad n \in \mathbb{N},\ \omega = \frac{\pi}{L}$$

Corresponding solutions: $\boxed{p_n(x) = \sin(n\omega x) = \sin\left(\frac{n\pi}{L}x\right)}$

We solve the second DE with these $\lambda-$values

$$\frac{\dot{q}_n(t)}{q_n(t)} = c\,\frac{p_n''(x)}{p_n(x)} = -\,c\cdot\lambda_n \iff \dot{q}_n(t) = -\,c\lambda_n q_n(t)$$

$$\Rightarrow \boxed{q_n(t) = e^{-c\lambda_n t} = e^{-c\omega^2 n^2 t}}$$

Each function $\boxed{\tilde{v}_n(t) = p_n(x)\cdot q_n(t) = \sin(n\omega x)\, e^{-c\omega^2 n^2 t}}$

fulfills the homogeneous DE and the homogeneous boundary values!

Homogeneous Heat Equation

Each linear combination $\alpha_n \tilde{v}_n + \alpha_m \tilde{v}_m$ fulfills the homogeneous DE and the homogeneous boundary values.

Proof: On the boundary it holds

$(\alpha_n \tilde{v}_n + \alpha_m \tilde{v}_m)(0) = \alpha_n \cdot \tilde{v}_n(0) + \alpha_m \cdot \tilde{v}_m(0) =$

$(\alpha_n \tilde{v}_n + \alpha_m \tilde{v}_m)(L) = \alpha_n \cdot \tilde{v}_n(L) + \alpha_m \cdot \tilde{v}_m(L) =$

Differential equation:

$$\begin{aligned}
(\alpha_n \tilde{v}_n + \alpha_m \tilde{v}_m)_t &- c\,(\alpha_n \tilde{v}_n + \alpha_m \tilde{v}_m)_{xx} \\
&= \alpha_n \cdot (\tilde{v}_n)_t + \alpha_m \cdot (\tilde{v}_m)_t - c\alpha_n (\tilde{v}_n)_{xx} - c\alpha_m (\tilde{v}_m)_{xx} \\
&= \alpha_n \left((\tilde{v}_n)_t - c(\tilde{v}_n)_{xx}\right) + \alpha_m \left((\tilde{v}_m)_t - c(\tilde{v}_m)_{xx}\right)
\end{aligned}$$

Question: Are linear combinations of solutions of inhomogeneous differential equations and/or inhomogeneous boundary values also solutions?

Homogeneous Heat Equation

Each finite linear combination

$$\boxed{\tilde{v}(x,t) = \textstyle\sum_{n=1}^{m} \alpha_n e^{-c\omega^2 n^2 t} \sin(n\omega x) \qquad \omega = \frac{\pi}{L}}$$

solves the DE and fulfills the boundary values.

Also the initial condition has to be fulfilled:

$$\tilde{v}(x,0) = \sum_{n=1}^{m} \alpha_n e^{-c\omega^2 n^2 \cdot 0} \sin(n\omega x) \overset{!}{=} v_0(x) \qquad\qquad x \in (0,L)$$

So $\boxed{\tilde{v}(x,0) = \sum_{n=1}^{m} \alpha_n \cdot \sin(n\omega x) \overset{!}{=} v_0(x) \qquad\qquad x \in (0,L)}$

This only holds for particular v_0.

Idea for arbitrary v_0: Consider the series (∞ instead of m) and choose: α_n as Fourier coefficients of the odd, $2L$-periodic continuation of v_0. (See next auditorium exercise). Here: particular v_0.

Inhomogeneous Heat Equation

with inhomogeneous BV and IV

$$
\begin{aligned}
u_t - cu_{xx} &= h(x,t), && c > 0,\ t > 0,\ x \in (a,b),\ \text{here } (0,L),\\
u(x,0) &= u_0(x), && x \in (a,b),\\
u(a,t) &= f(t), && t > 0,\\
u(b,t) &= g(t), && t > 0
\end{aligned}
$$

Inhomogeneous Heat Equation

with inhomogeneous BV and IV

$$
\begin{aligned}
u_t - cu_{xx} &= h(x,t), && c>0,\ t>0,\ x \in (a,b),\ \text{here } (0,L),\\
u(x,0) &= u_0(x), && x \in (a,b),\\
u(a,t) &= f(t), && t>0,\\
u(b,t) &= g(t), && t>0
\end{aligned}
$$

Method:

1. Homogenization (of the boundary values)
2. Solving the homogenized problem
3. Determine original solution

Inhomogeneous Heat Equation

1. Homogenization of the boundary values

Homogenization

$$v(x,t) := u(x,t) - \left[f(t) + \frac{x}{L}(g(t) - f(t))\right]$$

$$v(0,t) = u(0,t) - f(t) - \frac{0}{L}\left(g(t) - f(t)\right)$$

$$v(L,t) = u(L,t) - f(t) - \frac{L}{L}\left(g(t) - f(t)\right)$$

New DE for v:

$$u(x,t) := v(x,t) + f(t) + \frac{x}{L}\left(g(t) - f(t)\right)$$

$$u_t(x,t) = v_t(x,t) + \dot{f}(t) + \frac{x}{L}\left(\dot{g}(t) - \dot{f}(t)\right)$$

$$u_x(x,t) = v_x(x,t) + 0 + \frac{1}{L}\left(g(t) - f(t)\right)$$

$$u_{xx}(x,t) = v_{xx}(x,t)$$

Homogenized problem

$$
\begin{aligned}
v_t - cv_{xx} &= \tilde{h}(x,t), && c > 0,\ t > 0,\ x \in (0,L),\\
v(x,0) &= v_0(x), && x \in (0,L),\\
v(0,t) &= v(L,t) = 0, && t > 0
\end{aligned}
$$

with

$$\tilde{h}(x,t) = h(x,t) - f'(t) - \frac{x}{L}(g'(t) - f'(t))$$

and

$$v(x,0) = u(x,0) - f(0) - \frac{x}{L}(g(0) - f(0)) =: v_0(x).$$

This problem consists, in general, of an inhomogeneous DE, inhomogeneous initial values and **homogeneous boundary values**.

Inhomogeneous Heat Equation
2. Solve Homogenized Problem

$$\begin{aligned} v_t - cv_{xx} &= \tilde{h}(x,t), && c>0,\ t>0,\ x\in(0,L),\\ v(x,0) &= v_0(x), && x\in(0,L),\\ v(0,t) &= v(L,t)=0, && t>0 \end{aligned}$$

homogeneous boundary values are fulfilled by:
$p_n(x) = \sin(n\omega x) = \sin\left(\frac{n\pi}{L}x\right), \quad n\in\mathbb{N}$

Ansatz: $\boxed{v(x,t)=\sum_{n=1}^{m} a_n(t)\sin(n\omega x)}$

Plug into DE $v_t - cv_{xx} = \tilde{h}(x,t)$:

$$\sum_{n=1}^{m}\left[\dot{a}_n(t)+cn^2\omega^2 a_n(t)\right]\sin(n\omega x)=\tilde{h}(x,t)$$

The solution has to fulfill the initial value as well

$$v(x,0) = \sum_{n=1}^{m} a_n(0) \sin(n\omega x) = v_0(x)$$

Ideas for arbitrary $\tilde{h}, v_0$:
Consider the series (∞ instead of m) and replace the right handside by the Fourierseries of the odd, $2L$-periodic continuation of $\tilde{h}$ or v_0, respectively. We obtain an initial value problem with ordinary DEs for a_n.

Computation of Fourier coefficients follows in the next auditorium exercise.
Here: particular choices of $\tilde{h}, v_0$.

Inhomogeneous Heat Equation

3. Determine original solution

$$u(x,t) = v(x,t) + f(t) + \frac{x}{L}\left(g(t) - f(t)\right)$$

Inhomogeneous Heat Equation

Example

$$\begin{aligned}
u_t - u_{xx} &= \frac{x-\pi}{\pi\,(t+1)^2} + 4\sin(2x) && 0 < x < \pi,\ t \in \mathbb{R}^+, \\
u(x,0) &= 1 - \frac{x}{\pi} + \sin(6x) && 0 < x < \pi\,, \\
u(0,t) &= \frac{1}{t+1} && t > 0, \\
u(\pi,t) &= 0 && t > 0
\end{aligned}$$

1. Homogenization of boundary values

   $v(x,t) := u(x,t) - \left[f(t) + \frac{x}{L}(g(t) - f(t))\right]$

Inhomogeneous Heat Equation
Example

New DE for $v(x,t) = u(x,t) + \dfrac{1}{t+1}\left(\dfrac{x}{\pi} - 1\right)$

DE for u: $u_t - u_{xx} = \dfrac{x-\pi}{\pi\,(t+1)^2} + 4\sin(2x)$

$v_t(x,t) =$

$u_t - u_{xx} =$

$v_t - v_{xx} =$

$v(x,0) =$

$v(0,t) = u(0,t) - \dfrac{1}{t+1} =$

$v(\pi,t) = u(\pi,t) + \dfrac{1}{t+1}\left(\dfrac{\pi}{\pi} - 1\right) = 0.$

Inhomogeneous Heat Equation
Example

2. $v_t - v_{xx} = 4\sin(2x), \qquad x \in (0,\pi),\ t > 0$

$v(x,0) \ = \ \sin(6x) \qquad x \in [0,\pi]$

$v(0,t) \ = \ v(\pi,t) \ = \ 0, \qquad t \geq 0$

Ansatz:

$$v(x,t) = \sum_{n=1}^{m} a_n(t) \sin(n\omega x)$$

Plugging this into the PDE $v_t - cv_{xx} = \tilde{h}(x,t)$, yields

$$\sum_{n=1}^{m} [\dot{a}_n(t) + cn^2\omega^2 a_n(t)] \sin(n\omega x) = \tilde{h}(x,t)$$

2. continued

$$\sum_{n=1}^{m} [\dot{a}_n(t) + cn^2\omega^2 a_n(t)] \sin(n\omega x) = \tilde{h}(x,t)$$

where

$c = 1,\ \tilde{h}(x,t) = 4\sin(2x),\ L = \pi$ so $\omega = 1.$

So we have

$$\sum_{n=1}^{m} [\dot{a}_n(t) + n^2 a_n(t)] \sin(nx) = 4\sin(2x)$$

Comparison of coefficients yields the ordinary differential equations

Inhomogeneous Heat Equation

Example

2. continued

   The solution has to fulfill the initial values

$$v(x,0) = \sum_{n=1}^{m} a_n(0) \sin(nx) = v_0(x) = \sin(6x)$$

   Comparison of coefficients yields the initial values for our differential equations

   In total we get the following initial value problems

$$\dot{a}_n(t) + n^2 a_n(t) = 0,\ a_n(0) = 0 \qquad \forall n \notin \{2, 6\}$$

Inhomogeneous Heat Equation

Example

2. continued

   With solutions

   $a_n(t) =$

   and

   $\dot{a}_6(t) + 6^2 a_6(t) = 0, \;\; a_6(0) = 1$

   $\dot{a}_2(t) + 2^2 a_2(t) = 4, \;\; a_2(0) = 0$

   Solution: $v(x,t) =$

Inhomogeneous Heat Equation

Example

3.

$$u(x,t) = v(x,t) + f(t) + \frac{x}{L}\left(g(t) - f(t)\right)$$

where $f(t) = \frac{1}{1+t}$, $g(t) = 0$, $L = \pi$.

Inhomogeneous Heat Equation

Hint for Homework 2a

$$\frac{\dot{q}(t)}{q(t)} = c\,\frac{p''(x)}{p(x)} = \mu$$

So $\dot{q}(t) = \mu q(t)$

$x-$ part: linear, exponential or periodic.

Follow pages 5-9 with
$p'(0) = p'(L) = 0$ instead of $p(0) = p(L) = 0$.

Use the series instead of the sum, that is ∞ instead of m.

Laplace Equation on Circles, Arcs, inside and outside of Disks...

Laplace operator in polar coordinates:

$$\boxed{\Delta u = 0 \overset{r\neq 0}{\iff} u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\phi\phi} = 0\,.}$$

For **rotationally symmetric** data: $u(r,\phi) = w(r)$ it holds $u_{\phi\phi} = 0\,.$

We need to solve: $\qquad r^2 u_{rr} + r u_r = 0$

Let $g := w'$. Then we need to solve: $rg'(r) + g(r) = 0$

$$\frac{dg}{dr} = -\frac{g}{r} \iff \frac{dg}{g} = -\frac{dr}{r}$$

$$\ln(|g|) = -\ln(r) + k \iff e^{\ln(|g|)} = e^{-\ln(r)+k} = e^{-\ln(r)} \cdot e^k$$

$$w' = g = \alpha \cdot \frac{1}{r} \Longrightarrow w(r) = \alpha \cdot \ln(r) + \beta$$

Compare representation in terms of the fundamental solution from last auditorium exercise

Not Rotationally Symmetric Data

- **Ansatz:** $u(r, \phi) = w(r) \cdot v(\phi)$

  which should satisfy: $r^2 u_{rr} + r u_r + u_{\phi\phi} = 0\,.$

- New DE: $r^2 w'' \cdot v + r w' \cdot v + w \cdot v'' = 0$

- Order with respect to v and w:

$$v(r^2 w'' + r w') = -w \cdot v'' \Longrightarrow \frac{r^2 w'' + r w'}{w} = -\frac{v''}{v} = \lambda.$$

- System of ODEs:

$$\boxed{v''(\phi) = -\lambda v(\phi)\,, \qquad r^2 w''(r) + r w'(r) - \lambda w = 0}$$

- Solutions see above
- $\lambda = 0$: linear function
- $\lambda < 0$: real exp-functions
- $\lambda > 0$: Cos- and Sin functions

v should be $2\pi-$periodic such that it consists of $\cos(k\phi)$, $\sin(k\phi)$ with corresponding $\lambda_k = k^2$ and

$$\boxed{v_k(\phi) = a_k \cos(k\phi) + b_k \sin(k\phi)\,, \quad k \in \mathbb{N}, \quad v_0(\phi) = a_0}$$

According to the system, we get for $v = v_k$ the ODE for $w = w_k$

$$r^2 w''(r) + rw'(r) - \lambda_k w = r^2 w''(r) + rw'(r) - k^2 w = 0.$$

Not Rotationally Symmetric Data

Analysis of $r^2 w''(r) + rw'(r) - k^2 w = 0$

$k = 0$:

- $r^2 w''(r) + rw'(r) = 0$
- as above with $g = w'$
- $rg'(r) + g(r) = 0 \iff \dfrac{g'}{g} = -\dfrac{1}{r}$
- $w' = g = \dfrac{d_0}{r} \implies \boxed{w_0 = c_0 + d_0 \ln(r)\,.}$

$k \neq 0$:

- Euler's DE: Substitution $r = e^t$ or ansatz $w(r) = r^\gamma$

$$
\begin{aligned}
& r^2 w''(r) + rw'(r) - k^2 w = 0 \\
\iff & -k^2 \cdot r^\gamma + r \cdot \gamma \cdot r^{\gamma-1} + r^2 \cdot \gamma \cdot (\gamma - 1) \cdot r^{\gamma-2} = 0 \\
\iff & r^\gamma \left(-k^2 + \gamma + \gamma^2 - \gamma\right) = 0 \\
\iff & \gamma = \pm k
\end{aligned}
$$

- Thus, $\boxed{w_k(r) = c_k r^{-k} + d_k r^k}$

Not Rotationally Symmetric Data

Every function $w_k \cdot v_k$ solves the differential equation. Since the DE is linear, every linear combination is a solution as well:

$$u(r,\phi) = c_0 + d_0 \ln(r) + \sum_{k=1}^{n} (c_k r^{-k} + d_k r^k)(a_k \cos(k\phi) + b_k \sin(k\phi))$$

Without deeper analysis of the convergence, we write

$$\boxed{u(r,\phi) = c_0 + d_0 \ln(r) + \textstyle\sum_{k=1}^{\infty} (c_k r^{-k} + d_k r^k)(a_k \cos(k\phi) + b_k \sin(k\phi))}$$

Depending on the region, we need to restrict to bounded summands, see below.

Laplace equation on unbounded sets

Example

- $\Delta u = 0, \;$ for $x^2 + y^2 > 16, \qquad u(x, y) = 1 + xy - 2y^2, \;$ on $x^2 + y^2 = 16.$

- General solution:

$$u(r, \phi) = c_0 + d_0 \ln(r) + \sum_{k=1}^{\infty} (c_k r^{-k} + d_k r^k)(a_k \cos(k\phi) + b_k \sin(k\phi))$$

- Ansatz for $x^2 + y^2 > R^2$:
- Since the solution should be bounded: $d_k = 0\,, \quad \forall k.$
- It remains

$$u(r, \phi) = c_0 + \sum_{k=1}^{\infty} c_k r^{-k} (a_k \cos(k\phi) + b_k \sin(k\phi))$$

Laplace equation on unbounded sets

- Ansatz: $u(r,\phi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} r^{-k} \left(a_k \cos(k\phi) + b_k \sin(k\phi)\right)$
- Boundary conditions: $u(R,\phi) = u_R(\phi)$, thus

$$u(R,\phi) = \frac{a_0}{2} + \sum_{k=1}^{\infty} R^{-k} \left(a_k \cos(k\phi) + b_k \sin(k\phi)\right) = u_R(\phi)$$

- Develop u_R with a Fourier series

$$u_R(\phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} A_k \cos(k\phi) + B_k \sin(k\phi)$$

$$A_k = \frac{1}{\pi} \int_0^{2\pi} u_R(\phi) \cos(k\phi)\, d\phi, \qquad B_k = \frac{1}{\pi} \int_0^{2\pi} u_R(\phi) \sin(k\phi)\, d\phi$$

- Compare coefficients:

$$R^{-k} a_k = A_k \iff a_k = R^k \cdot A_k, \quad \text{and analogously } b_k = R^k \cdot B_k$$

such that we obtain the solution for the complement

$$u(r,\phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{R}{r}\right)^k \left(A_k \cos(k\phi) + B_k \sin(k\phi)\right)$$

Computation of Fourier coefficients: next auditorium exercise

Laplace equation on unbounded sets

Example

$$u(r,\phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{R}{r}\right)^k \left(A_k \cos(k\phi) + B_k \sin(k\phi)\right)$$

- For our example, we have the boundary values $u(x,y) = 1 + xy - 2y^2$, on $x^2 + y^2 = 16$
- With $x = r\cos(\phi)$, $y = r\sin(\phi)$, $r^2 = x^2 + y^2$ this is

$$u(4,\phi) = 1 + 4\cos(\phi) \cdot 4\sin(\phi) - 32\sin^2(\phi)$$

- Use $\sin(2\phi) = 2\cos(\phi)\cdot\sin(\phi)$, $\cos(2\phi) = 1 - 2\sin^2(\phi)$
- Then,

$$u(4,\phi) = u_R(\phi) = 1 + 8\sin(2\phi) - 16 + 16\cos(2\phi)$$
$$\stackrel{!}{=} \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{4}{4}\right)^k \left(A_k \cos(k\phi) + B_k \sin(k\phi)\right)$$

$$u(4,\phi) = u_R(\phi) = 1 + 8\sin(2\phi) - 16 + 16\cos(2\phi)$$
$$\stackrel{!}{=} \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{4}{4}\right)^k (A_k\cos(k\phi) + B_k\sin(k\phi))$$

- The Fourier coefficients of u_0 are

  $$\frac{A_0}{2} = \quad , \quad A_2 = \quad , \quad B_2 = \quad , \quad A_k = B_k = \quad \text{else}$$

  (In general one has to determine the Fourier coefficients by integrals.)

- With this we get the solution

  $u(r,\phi) =$

Laplace equation on unbounded sets

Example

- We can reformulate it as a solution depending on $x, y \in \mathbb{R}^2$ (Cartesian coordinates). Use $\cos(\phi) = \frac{x}{r}$, $\sin(\phi) = \frac{y}{r}$.
- $$u(r,\phi) = -15 + \left(\frac{4}{r}\right)^2 \left(16\cos(2\phi) + 8\sin(2\phi)\right)$$
- $\cos(2\phi) = \cos^2(\phi) - \sin^2(\phi) =$
- $\sin(2\phi) = 2\cos(\phi)\cdot\sin(\phi) =$
- $u(r,\phi) = -15 + \left(\frac{4}{r}\right)^2 \left(16\cos(2\phi) + 8\sin(2\phi)\right)$
- $u(x,y) = -15 + \left(\frac{4}{\sqrt{x^2+y^2}}\right)^2 \left(16\frac{x^2-y^2+xy}{x^2+y^2}\right)$

Summary

- General ansatz for (complement resp. sections of) circles or rings:

$$\boxed{u(r,\phi) = c_0 + d_0 \ln(r) + \sum_{k=1}^{\infty} (c_k r^{-k} + d_k r^k)(a_k \cos(k\phi) + b_k \sin(k\phi))}$$

- For getting bounded solutions one sets
  - On **balls** with BV $u(R,\phi) = u_R(\phi)$: $c_k = 0, \forall k \in \mathbb{N}$ and $d_0 = 0$:

$$u(r,\phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k (A_k \cos(k\phi) + B_k \sin(k\phi))$$

  - On **complements of balls** with BV $u(R,\phi) = u_R(\phi)$: $d_k = 0$ and:

$$u(r,\phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{R}{r}\right)^k (A_k \cos(k\phi) + B_k \sin(k\phi))$$

  - In **rings** with $u(R_1,\phi) = u_1(\phi)$, $\quad u(R_2,\phi) = u_2(\phi)$: Use complete ansatz. Determine coefficients via two boundary conditions.
  - In **segments of a ring**: If there are boundary values $\neq 0$ on more than one boundary part, one may need to disassemble the region. It may also be necessary to adjust the ansatz.