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Announcement

No exercise class on monday

Auditorium Exercise 04

Differential Equations II for Students of Engineering Sciences
Summer Semester 2025

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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

Overview

Second Order Differential Equations

Harmonic Functions

Laplace Equation

Linear Second Order Differential Equations

Here just two variables. General case: see lecture notes.

$$\boxed{a(x,t)u_{xx} + 2b(x,t)u_{xt} + c(x,t)u_{tt} \;=\; h(x,t,u,u_x,u_t)}$$

Classification:

$D(x,t) = \; a(x,t)c(x,t) - (b(x,t))^2 < 0 \; : \;$ hyperbolic,
$D(x,t) = \; a(x,t)c(x,t) - (b(x,t))^2 = 0 \; : \;$ parabolic,
$D(x,t) = \; a(x,t)c(x,t) - (b(x,t))^2 > 0 \; : \;$ elliptic.

Different methods are suitable, different initial or boundary conditions are required for reasonable * problems.

*) Reasonable means well defined: There exists a unique solution, depending continuously on the initial and/or boundary data.

Examples

Compare HW1

Determine the type of the following differential equations:

1. $u_{xx} + 6\, u_{xy} + 9\, u_{yy} + \; y\, u_x - x\, u_y \;=\; 0$

2. $(2x^2 - 1)\, u_{xx} - 4xy\, u_{xy} + (2y^2 - 1)\, u_{yy} + x\, u_x \;=\; \cos(x)$

Linear PDE of second order with constant coefficients

(derivatives of second order)

$$a_{11}u_{xx} + 2a_{12}u_{xt} + a_{22}u_{tt} + b_1(x,t)u_x + b_2(x,t)u_t + c(x,t)u = h(x,t)$$

Diagonal form: No mixed derivatives of second order

Normal form: No mixed derivatives of second order and coefficients of second derivatives take values $\in \{-1, 0, 1\}$.

Integrable form: Only mixed derivatives of second order

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Normal form: No mixed derivatives of second order and coefficients of second derivatives take values $\in \{-1, 0, 1\}$.

Integrable form: Only mixed derivatives of second order

hyperbolic: $$u_{xx} - u_{tt} = G(x, t, u, u_x, u_t)$$

or $$u_{xt} = G(x, t, u, u_x, u_t)$$

parabolic: $$u_{xx} = G(x, t, u, u_x, u_t)$$

elliptic: $$u_{xx} + u_{tt} = G(x, t, u, u_x, u_t)$$

Goal: Derive diagonal/normal/integrable form

We introduce new variables $\eta = \eta(x,t)$, $\tau = \tau(x,t)$,

and a function: $v(\eta(x,t), \tau(x,t)) = u(x,t)$

Regularity condition: $\eta_x \tau_t - \eta_t \tau_x \neq 0$.

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We use the chain rule to compute the derivatives:

$u_x = (v(\eta(x, t), \tau(x, t)))_x =$

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$$u_x = \left(v(\eta(x,t), \tau(x,t))\right)_x =$$

$$u_t = v_\eta \cdot \eta_t + v_\tau \cdot \tau_t \,,$$

$$u_{xx} =$$

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We use the chain rule to compute the derivatives:

$$u_x = \left(v(\eta(x,t), \tau(x,t))\right)_x =$$

$$\begin{aligned}
u_t &= v_\eta \cdot \eta_t + v_\tau \cdot \tau_t\,, \\
u_{xx} &= \\
&= v_{\eta\eta} \cdot (\eta_x)^2 + 2v_{\eta\tau} \cdot \eta_x \tau_x + v_{\tau\tau} \cdot (\tau_x)^2 + \left(v_\eta \eta_{xx} + v_\tau \tau_{xx}\right), \\
u_{xt} &= v_{\eta\eta} \cdot \eta_x \eta_t + v_{\eta\tau} \cdot (\tau_t \eta_x + \tau_x \eta_t) + v_{\tau\tau} \cdot \tau_t \tau_x + \left(v_\eta \eta_{xt} + v_\tau \tau_{xt}\right), \\
u_{tt} &= v_{\eta\eta} \cdot (\eta_t)^2 + 2v_{\eta\tau} \cdot \eta_t \tau_t + v_{\tau\tau} \cdot (\tau_t)^2 + \left(v_\eta \eta_{tt} + v_\tau \tau_{tt}\right).
\end{aligned}$$

We plug this in and obtain another differential equation

$$Av_{\eta\eta} + 2Bv_{\eta\tau} + Cv_{\tau\tau} = \tilde{h}(\eta, \tau, v, v_\eta, v_\tau)$$

Question: How to choose η, τ?

- Normal form: see lecture notes. Here: a short sketch of the method without example.
- Example for integrable form follows.

Example for Exercise 1 (Work sheet): Transform to integrable form

For $$u_{tt} + (a+b)u_{tx} + abu_{xx} = 0$$
substitute $\alpha = x - bt$, $\mu = x - at$.

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For $$u_{tt} + (a+b)u_{tx} + abu_{xx} \;=\; 0$$
substitute $\alpha = x - bt$, $\mu = x - at$.

Example:

$$14u_{xx} + 9u_{xt} + u_{tt} = \;0 \quad \text{for } x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = u_0(x), \quad \text{for } x \in \mathbb{R} \qquad \text{z.B.: } u_0(x) = x + \sin(x),$$

$$u_t(x,0) = v_0(x), \quad \text{for } x \in \mathbb{R} \qquad \text{z.B.:} v_0(x) = -7 - 2\cos(x).$$

$$\alpha \;=\; x - 7t,\ \mu \;=\; x - 2t$$
$$u(x,t) = v(\alpha(x,t), \mu(x,t))$$

Formulas given on page 8 with α, μ instead of η and τ

$$u_x = \left(v(\alpha(x,t), \mu(x,t))\right)_x =$$

$$u_x = \left(v(\alpha(x,t), \mu(x,t))\right)_x =$$

$$u_t = v_\alpha \cdot \alpha_t + v_\mu \cdot \mu_t$$

$$u_{xx} = v_{\alpha\alpha} \cdot (\alpha_x)^2 + 2v_{\alpha\mu} \cdot \alpha_x \mu_x + v_{\mu\mu} \cdot (\mu_x)^2 + \left(v_\alpha \alpha_{xx} + v_\mu \mu_{xx}\right),$$

$$u_{xt} = v_{\alpha\alpha} \cdot \alpha_x \alpha_t + v_{\alpha\mu} \cdot (\mu_t \alpha_x + \mu_x \alpha_t) + v_{\mu\mu} \cdot \mu_t \mu_x + (v_\alpha \alpha_{xt} + v_\mu \mu_{xt}),$$

$$u_{tt} = v_{\alpha\alpha} \cdot (\alpha_t)^2 + 2v_{\alpha\mu} \cdot \alpha_t \mu_t + v_{\mu\mu} \cdot (\mu_t)^2 + \left(v_\alpha \alpha_{tt} + v_\mu \mu_{tt}\right).$$

$$u_x = \left(v(\alpha(x,t),\mu(x,t))\right)_x =$$

$$u_t = v_\alpha \cdot \alpha_t + v_\mu \cdot \mu_t$$

$$u_{xx} = v_{\alpha\alpha} \cdot (\alpha_x)^2 + 2v_{\alpha\mu} \cdot \alpha_x \mu_x + v_{\mu\mu} \cdot (\mu_x)^2 + \left(v_\alpha \alpha_{xx} + v_\mu \mu_{xx}\right),$$

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Plug into differential equation $14u_{xx} + 9u_{xt} + u_{tt} = 0$

$$u_x = \left(v(\alpha(x,t),\mu(x,t))\right)_x =$$

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Plug into differential equation $14u_{xx} + 9u_{xt} + u_{tt} = 0$

New differential equation $v_{\alpha\mu} = 0.$

Alternative

For linear transformations and the following differential equation

$$a_{11} u_{xx} + 2a_{12} u_{xt} + a_{22} u_{tt} + b_1(x,t) u_x + b_2(x,t) u_t + c(x,t) u = h(x,t)$$

Principal part: $\left(a_{11} \frac{\partial}{\partial x}\frac{\partial}{\partial x} + 2a_{12}\frac{\partial}{\partial x}\frac{\partial}{\partial t} + a_{22}\frac{\partial}{\partial t}\frac{\partial}{\partial t}\right) u$

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Matrix notation:
$(\nabla^T \boldsymbol{A} \nabla)u$

Rewriting the first derivatives analogously yields:

$$(\nabla^T \boldsymbol{A} \nabla)u + (b^T \nabla)u + cu = h, \qquad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

Our differential equation: $14u_{xx} + 9u_{xt} + u_{tt} = 0$

$$\iff (\nabla^T \boldsymbol{A} \nabla) u \;=\; \nabla^T \begin{pmatrix} 14 & \frac{9}{2} \\ \frac{9}{2} & 1 \end{pmatrix} \nabla \cdot u = \mathbf{0}$$

$$\Longleftrightarrow (\nabla^T \boldsymbol{A} \nabla) u \;=\; \nabla^T \begin{pmatrix} 14 & \frac{9}{2} \\ \frac{9}{2} & 1 \end{pmatrix} \nabla \cdot u = \mathbf{0}$$

Transformation: $\alpha \;=\; x - 7t,\; \mu \;=\; x - 2t$

$$\begin{pmatrix} \alpha \\ \mu \end{pmatrix} = \begin{pmatrix} 1 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} x \\ t \end{pmatrix} =: \boldsymbol{S}^T \begin{pmatrix} x \\ t \end{pmatrix}$$

From above:

$$u_x = \; 1 \cdot v_\alpha + 1 \cdot v_\mu \,,$$
$$u_t = -7 \cdot v_\alpha - 2 \cdot v_\mu$$

$$\iff (\nabla^T \boldsymbol{A} \nabla) u \;=\; \nabla^T \begin{pmatrix} 14 & \frac{9}{2} \\ \frac{9}{2} & 1 \end{pmatrix} \nabla \cdot u = \mathbf{0}$$

Transformation: $\quad \alpha \;=\; x - 7t,\; \mu \;=\; x - 2t$

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From above:
$$u_x = \; 1 \cdot v_\alpha + 1 \cdot v_\mu \,,$$
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$$\nabla_{xt} \cdot \boldsymbol{u} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} \cdot \boldsymbol{u} = \begin{pmatrix} u_x \\ u_t \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ -7 & -2 \end{pmatrix} \begin{pmatrix} v_\alpha \\ v_\mu \end{pmatrix} = \boldsymbol{S} \begin{pmatrix} v_\alpha \\ v_\mu \end{pmatrix} = \; \boldsymbol{S} \nabla_{\alpha\mu} \cdot \boldsymbol{v}$$

Therefore $\quad \nabla_{xt} = \begin{pmatrix} \frac{\partial}{\partial x} \\ \frac{\partial}{\partial t} \end{pmatrix} = \boldsymbol{S} \cdot \nabla_{\alpha\mu} = \boldsymbol{S} \cdot \begin{pmatrix} \frac{\partial}{\partial \alpha} \\ \frac{\partial}{\partial \mu} \end{pmatrix}$

$$(\nabla_{xt}^T \boldsymbol{A} \nabla_{xt}) \boldsymbol{u} = \; (\, \boldsymbol{S} \cdot \nabla_{\alpha\mu} \,)^T \cdot \boldsymbol{A} \cdot \boldsymbol{S} \cdot \nabla_{\alpha\mu} \, \boldsymbol{v} = \; \nabla_{\alpha\mu}^T \, (\boldsymbol{S}^T \cdot \boldsymbol{A} \cdot \boldsymbol{S}) \, \nabla_{\alpha\mu} \, \boldsymbol{v} \;=\; 0$$

Here it holds that:

$$\boldsymbol{S}^T\boldsymbol{A}\boldsymbol{S} = \begin{pmatrix} 1 & -7 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} 14 & \frac{9}{2} \\ \frac{9}{2} & 1 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ -7 & -2 \end{pmatrix} = \begin{pmatrix} 0 & -\frac{25}{2} \\ -\frac{25}{2} & 0 \end{pmatrix}$$

New differential equation:

$$\nabla_{\alpha\mu}^T \boldsymbol{S}^T\boldsymbol{A}\boldsymbol{S}\nabla_{\alpha\mu}\, \boldsymbol{v} = \left(\tfrac{\partial}{\partial\alpha},\, \tfrac{\partial}{\partial\mu}\right) \begin{pmatrix} 0 & -\frac{25}{2} \\ -\frac{25}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\alpha} \\ \frac{\partial}{\partial\mu} \end{pmatrix} = -25\boldsymbol{v}_{\alpha\mu} = 0$$

Here it holds that:

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New differential equation:

$$\nabla_{\alpha\mu}^T \boldsymbol{S}^T\boldsymbol{A}\boldsymbol{S}\nabla_{\alpha\mu}\,\boldsymbol{v} = (\tfrac{\partial}{\partial\alpha},\ \tfrac{\partial}{\partial\mu}) \begin{pmatrix} 0 & -\frac{25}{2} \\ -\frac{25}{2} & 0 \end{pmatrix} \begin{pmatrix} \frac{\partial}{\partial\alpha} \\ \frac{\partial}{\partial\mu} \end{pmatrix} = -25\boldsymbol{v}_{\alpha\mu} = 0$$

$\boldsymbol{v}_{\alpha\mu} = 0 \implies$

$\boldsymbol{v}_{\alpha} = \qquad\qquad \implies$

$\boldsymbol{v}(\alpha,\mu) = \Phi(\alpha) + \Psi(\mu) \implies u(x,t) =$

Determine Φ and Ψ by the initial conditions:

$u(x,0) = u_0(x)$

$u_t(x,0) = v_0(x)$

For our example with

$$u(x,t) = \Phi(x-7t) + \Psi(x-2t)$$

and

$$u_0(x) = x + \sin(x),\ v_0(x) = -7 - 2\cos(x)$$

we obtain the conditions

$$u(x,0) = \Phi(x) + \Psi(x) = x + \sin(x)$$

and

$v(x,0) =$

Transformation to normal form

See lecture notes. Here: brief sketch

For linear partial differential equations of second order with constant coefficients

$$a_{11}u_{xx} + 2a_{12}u_{xt} + a_{22}u_{tt} + b_1(x,t)u_x + b_2(x,t)u_t + c(x,t)u = h(x,t)$$

Matrix notation:

$$(\nabla^T \boldsymbol{A} \nabla)u + (b^T\nabla)u + cu = h, \qquad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

A ist real and symmetric:
Determine eigenvalues λ_1, λ_2 and corresponding orthonormal eigenvectors $\boldsymbol{v}^{[1]}, \boldsymbol{v}^{[2]}$.

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See lecture notes. Here: brief sketch

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$$a_{11} u_{xx} + 2a_{12} u_{xt} + a_{22} u_{tt} + b_1(x,t) u_x + b_2(x,t) u_t + c(x,t) u = h(x,t)$$

Matrix notation:

$$(\nabla^T \boldsymbol{A} \nabla) u + (b^T \nabla) u + cu = h, \qquad \boldsymbol{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{12} & a_{22} \end{pmatrix}$$

A ist real and symmetric:
Determine eigenvalues λ_1, λ_2 and corresponding orthonormal eigenvectors $\boldsymbol{v}^{[1]}, \boldsymbol{v}^{[2]}$. Let $\boldsymbol{S} = (\boldsymbol{v}^{[1]}, \boldsymbol{v}^{[2]})$, $\begin{pmatrix} \eta \\ \tau \end{pmatrix} = \boldsymbol{S}^T \begin{pmatrix} x \\ t \end{pmatrix}$
Change of coordinates gives the **diagonal form**:

$$\lambda_1 v_{\eta\eta} + \lambda_2 v_{\tau\tau} + p_1 v_\eta + p_2 v_\tau + dv = H$$

Hyperbolic: $\lambda_1 \cdot \lambda_2 < 0$, Elliptic: $\lambda_1 \cdot \lambda_2 > 0$, Parabolic: $\lambda_1 \cdot \lambda_2 = 0$.

For hyperbolic and elliptic equations scaling

$\hat{x} = \eta/\sqrt{\lambda_1}$, $\hat{t} = \tau/\sqrt{\lambda_2}$ gives the **normal forms**

$$\hat{u}_{\hat{x}\hat{x}} \pm \hat{u}_{\hat{t}\hat{t}} + p_1\hat{u}_{\hat{x}} + p_2\hat{u}_{\hat{t}} + d\hat{u} = H$$

For parabolic equations one eigenvalue, e.g., λ_2, is zero. One of the second derivatives is missing, e.g.. $\tilde{u}_{\tau\tau}$. We divide the diagonal form by λ_1.

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Examples Diagonal Forms

Hyperbolic: Wave equation $u_{tt} - c^2 u_{xx} = 0$,
Elliptic: Potential-/Laplace equation $\Delta u = u_{xx} + u_{yy} = 0$,
Parabolic: Heat equation $u_t - c u_{xx} = 0$.

Harmonic functions

Definition

Let $\Omega \subset \mathbb{R}^n$ be a bounded connected and open region with boundary $\partial\Omega$. A function $u \in C^2(\Omega) \cap C(\Omega \cup \partial\Omega)$ is called **harmonic** in Ω, if

$$\Delta u(x) = \sum_{i=1}^{n} u_{x_i x_i}(x) = 0 \qquad \forall\, x \in \Omega$$

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$$\Delta u(x) = \sum_{i=1}^{n} u_{x_i x_i}(x) = 0 \qquad \forall\, x \in \Omega$$

Example (Compare WS2a)

For which $k \in \mathbb{R}$ and $g : \mathbb{R} \to \mathbb{R}$ is u harmonic in $\mathbb{R}^2$?

$$\Delta u = u_{xx} + u_{yy} \stackrel{!}{=} 0 \qquad \forall\, (x, y) \in \mathbb{R}^2$$

a) $u(x, y) := (x + k \cdot y)^2$

b) $u(x, y) := e^{-3x} \cdot g(y)$

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a) $u(x, y) := (x + k \cdot y)^2$

b) $u(x, y) := e^{-3x} \cdot g(y)$

$$u_x = -3e^{-3x} \cdot g(y), \qquad u_y = e^{-3x} \cdot g'(y)$$
$$u_{xx} = 9\, e^{-3x} \cdot g(y), \qquad u_{yy} = e^{-3x} \cdot g''(y)$$
$$\begin{aligned} \Delta u(x, y) &= u_{xx}(x, y) + u_{yy}(x, y) \\ &= 9\, e^{-3x} \cdot g(y) + e^{-3x} \cdot g''(y) \\ &= e^{-3x} \left(g''(y) + 9g(y)\right) \overset{!}{=} 0 \end{aligned}$$

Properties of Harmonic Functions

Theorem (Mean Value Property)

Let u be harmonic in $B_a(x_0, y_0)$ and continuously extendable up on the boundary $\partial B_a(x_0, y_0)$ of this ball. Then it holds

$$u(x_0, y_0) = \frac{1}{2\pi a} \oint_{\partial B_a(x_0, y_0)} u(x, y) dS$$

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Theorem (Maximum Principle)

Let Ω be as before. Any harmonic function u on Ω reaches its max or min on the boundary of Ω, i.e. $\max_\Omega u = \max_{\partial\Omega} u$ and $\min_\Omega u = \min_{\partial\Omega} u$.

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Let Ω be as before. Any harmonic function u on Ω reaches its max or min on the boundary of Ω, i.e. $\max_\Omega u = \max_{\partial\Omega} u$ and $\min_\Omega u = \min_{\partial\Omega} u$.

Theorem (Uniqueness)

Let Ω be now only bounded and open and $f \in \mathrm{C}^0(\Omega)$, $g \in \mathrm{C}^0(\partial\Omega)$. Then there exists at most one solution $u \in \mathrm{C}^2(\Omega) \cap \mathrm{C}^0(\overline{\Omega})$ to the BVP

$$\triangle u = f \quad \text{in } \Omega, \qquad u = g \quad \text{on } \partial\Omega.$$

Solution Formula to the Laplace Equation

Poisson's integral formula

Consider the Laplace BVP

$$u_{xx} + u_{yy} = 0, \qquad x, y \in B_R(x_0, y_0)$$
$$\text{i.e.}(x - x_0)^2 - (y - y_0)^2 < R^2$$
$$u_{xx} + u_{yy} = g(x, y), \qquad x, y \in \partial B_R(x_0, y_0)$$

Solution Formula to the Laplace Equation

Poisson's integral formula

Consider the Laplace BVP

$$
\begin{aligned}
u_{xx} + u_{yy} = 0, \qquad & x, y \in B_R(x_0, y_0) \\
& \text{i.e.} (x - x_0)^2 - (y - y_0)^2 < R^2 \\
u_{xx} + u_{yy} = g(x, y), \qquad & x, y \in \partial B_R(x_0, y_0)
\end{aligned}
$$

Then the solution is given by

$$
u(x, y) = \frac{R^2 - (x - x_0)^2 - (y - y_0)^2}{2\pi R} \int_{\|z-(x_0,y_0)^T\|=R} \frac{g(z)}{\|z - (x, y)^t\|^2} \, dz
$$

Fundamental Solution

Consider the Laplace BVP

$$
\begin{aligned}
u_{xx} + u_{yy} &= 0, \qquad x, y \in B_R(x_0, y_0) \\
u_{xx} + u_{yy} &= g(x, y), \qquad x, y \in \partial B_R(x_0, y_0)
\end{aligned}
$$

Fundamental Solution

Consider the Laplace BVP

$$u_{xx} + u_{yy} = 0, \qquad x, y \in B_R(x_0, y_0)$$
$$u_{xx} + u_{yy} = g(x, y), \qquad x, y \in \partial B_R(x_0, y_0)$$

We define the **fundamental solution**

$n = 2$ $\Phi(x) = \frac{1}{2\pi} \ln(\|x\|_2)$

$n > 2$ $\Phi(x) = -\frac{1}{\alpha_n (n-2) n}(\|x\|_2^{2-n})$

$\alpha_n =$ volume of unit ball in $\mathbb{R}^n$

Fundamental Solution

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Every **rotationally symmetric** harmonic function on $\mathbb{R}^n \setminus \{0\}$ has the form

$$u(x) = a\Phi(x) + b$$

for some $a, b \in \mathbb{R}$ (see lecture notes p. 61-63).

Example (Compare WS2b, HW2)

We are looking for the value $u(1,2)^T$ of the C^2 function which satisfies

$$u_{xx} + u_{yy} = 0 \qquad \text{for} \quad (x-1)^2 + (y-2)^2 < 9, \qquad \text{and} \ldots$$

a) $\ldots u(x,y) = 2025 \qquad \text{for} \quad (x-1)^2 + (y-2)^2 = 9.$

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b) $\ldots u(x,y) = x^2 - y^2 + 2025 \qquad \text{for} \quad (x-1)^2 + (y-2)^2 = 9.$

Hint: $\cos(2t) = \cos^2(t) - \sin^2(t).$

Method 1

Let K denote the disk with radius $r = 3$ around $(1, 2)^T$ (lecture $B_3(1, 2)$)
and

$$\mathbf{c}(t) = (1 + 3\cos(t),\, 2 + 3\sin(t))^T,\ t \in [0, 2\pi]$$

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a parametrization of ∂K (lecture $\partial B_3(1,2)$).

The mean value property then yields

$$u(1,2) = \frac{1}{2\pi r} \int_{\partial K} u(x,y)\, d(x,y) = \frac{1}{2\pi \cdot r} \int_0^{2\pi} u(\mathbf{c}(t)) \cdot \|\dot{\mathbf{c}}(t)\|\, dt$$

$$u(1,2) = \frac{1}{2\pi r} \int_{\partial K} u(x,y)\, d(x,y) = \frac{1}{2\pi \cdot 3} \int_{\partial K} (x^2 - y^2 + 2025)\, d(x,y)$$

$$= \frac{1}{6\pi} \int_0^{2\pi} u(\mathbf{c}(t)) \cdot \|\dot{\mathbf{c}}(t)\|\, dt$$

$$= \frac{1}{6\pi} \int_0^{2\pi} ((1 + 3\cos(t))^2 - (2 + 3\sin(t))^2 + 2025) \cdot 3\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (1 + 6\cos(t) + 9\cos^2(t) - 4 - 12\sin(t) - 9\sin^2(t) + 2025)\, dt$$

$$= \frac{1}{2\pi} \int_0^{2\pi} (9\cos(2t) - 3 + 2025)\, dt$$

$$= 2022$$

Method 2 Poisson's integral formula

$$u(x,y) = \frac{R^2 - (x-1)^2 - (y-2)^2}{2\pi R} \int_{\|z - \binom{1}{2}\| = R} \frac{g(\boldsymbol{z})}{\|\boldsymbol{z} - \binom{x}{y}\|^2} \, d\boldsymbol{z}$$

$$u(1,2) = \frac{3^2 - 0^2 - 0^2}{2\pi 3} \int_{\|z - \binom{1}{2}\| = 3} \frac{z_1^2 - z_2^2 + 2025}{\|\boldsymbol{z} - \binom{1}{2}\|^2} \, d\boldsymbol{z}$$

$$= \frac{9}{6\pi} \int_0^{2\pi} \frac{((1 + 3\cos(t))^2 - (2 + 3\sin(t))^2 + 2025)}{3^2} \cdot \|\dot{\mathbf{c}}(t)\| \, dt$$

$$= \frac{3}{6\pi} \int_0^{2\pi} (1 + 6\cos(t) + 9\cos^2(t) - 4 - 12\sin(t) - 9\sin^2(t) + 2025) \, dt$$

$$= 2022 \,.$$

Method 3 The solution is unique. Due to

$$(x^2-y^2+2025)_{xx}+(x^2-y^2+2025)_{yy} = 2-2 = 0, \qquad \forall x, y \in \mathbb{R}$$

we already know the solution $u(x, y) = x^2 - y^2 + 2025$. Thus, $u(1, 2) = 1^2 - 2^2 + 2025 = 2022$.

Rotational symmetric solutions

According to the lecture every rotational symmetric harmonic function on $\mathbb{R}^n \backslash \{0\}$ can be expressed in terms of the fundamental solution $\Phi(x)$ as

$$u(x) = a\Phi(x) + b, \qquad a, b \in \mathbb{R}.$$

In the case of **symmetric** data (Ω, $0 \notin \Omega$ and boundary values), use fundamental solutions.

Example (Compare WS3)

Determine a solution to

$$\begin{aligned} \Delta v &= 0 \quad \text{for } e^2 < x^2 + y^2 < e^6, \\ v(x,y) &= 1 \quad \text{on } x^2 + y^2 = e^2, \\ v(x,y) &= 0 \quad \text{on } x^2 + y^2 = e^6. \end{aligned}$$

The fundamental solution

$$\Phi(x, y) = -\frac{1}{2\pi} \ln(\|(x, y)\|_2)$$

yields the solutions

$$v(x, y) = a\Phi(x, y) + b$$

for the PDE.

The fundamental solution

$$\Phi(x,y) = -\frac{1}{2\pi}\ln(\|(x,y)\|_2)$$

yields the solutions

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for the PDE.

Let

$$v(x,y) = u(r,\phi) = w(r), \qquad r = \sqrt{x^2+y^2}.$$

This results in

$$u(r,\phi) = w(r) = -\frac{a}{2\pi}\ln(r) + b.$$

The boundary values give

$$u(e^1, \phi) = 1 \implies -\frac{a}{2\pi}\ln(e^1) + b = 1$$
$$\implies -\frac{a}{2\pi} + b = 1 \implies b = 1 + \frac{a}{2\pi}.$$

The boundary values give

$$
\begin{aligned}
u(e^1,\phi)=1 &\implies -\frac{a}{2\pi}\ln(e^1)+b=1\\
&\implies -\frac{a}{2\pi}+b=1 \implies b=1+\frac{a}{2\pi}.\\
u(e^3,\phi)=0 &\implies -\frac{a}{2\pi}\ln(e^3)+1+\frac{a}{2\pi}=0 \implies\\
&\implies a=\pi, \qquad b=1+\frac{\pi}{2\pi}
\end{aligned}
$$

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$$u(e^1,\phi)=1 \implies -\frac{a}{2\pi}\ln(e^1)+b=1$$

$$\implies -\frac{a}{2\pi}+b=1 \implies b=1+\frac{a}{2\pi}.$$

$$u(e^3,\phi)=0 \implies -\frac{a}{2\pi}\ln(e^3)+1+\frac{a}{2\pi}=0 \implies$$

$$\implies a=\pi, \qquad b=1+\frac{\pi}{2\pi}$$

$$u(r,\phi)=-\frac{\pi}{2\pi}\ln(r)+\frac{3}{2}.$$

$$v(x,y)=\frac{3-\ln\left(\sqrt{x^2+y^2}\right)}{2}.$$