# Disclaimer

This version is based on the German original:

**Hörsaalübung zu Blatt 3 Differentialgleichungen II Erhaltungsgleichugen; Stoß- und Verdünungswellen, Burgers Gleichung**

created by Dr. Hanna Peywand Kiani

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# Auditorium Exercise 04

## Differential Equations II for Students of Engineering Sciences
## Summer Semester 2024


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Universität Hamburg
DER FORSCHUNG | DER LEHRE | DER BILDUNG

# Conservation Laws

# Continuity Equation: Reminder

- $u(x,t)$: density
- $q(x,t)$: flow
- $v(x,t)$: velocity
- $M(t) = \int_\Omega u(x,t)dx$ mass
- also

$$M(t) = M(t_0) + \int_{t_0}^{t} \left( \int_{\partial\Omega} q(x,t) \cdot n(x,t) dx \right) dt$$

  comp. HW 2
- By some computation and regularity assumptions, we got

$$u_t(x,t) + \nabla q(x,t) = 0$$

# Recap: Methods of Characteristics

## Basic idea

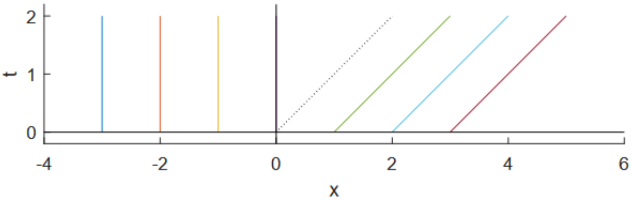
For a given PDE find curves (characteristics) $\gamma$ where the solution to the problem is constant and arrange the information to a full solution.

## Evolved idea

Reduce PDEs to underlying ODEs on characteristics in order to find solutions via known ODE-methods.

## Today

This does not have give full or unique solutions.



# Scalar Conservation Laws

## General space dimension

$u_t + \mathrm{div}_x(F(u)) \equiv 0, \qquad \text{in } (0, T) \times \Omega.$

## Space dimension 1

$u_t + f(u)\, u_x \equiv 0, \qquad \text{in } (0, \infty) \times \mathbb{R}.$

## Example

Easiest case: $f \equiv const.$ gives the linear transport equation.

## Today: Burgers equation

$f(u) = u$ gives

$$u_t + \frac{1}{2}\left(u^2\right)_x = u_t + u\, u_x \equiv 0, \qquad \text{in } (0, \infty) \times \mathbb{R}.$$

# Burger's Equation

$$
\begin{aligned}
u_t + u\,u_x &\equiv 0, && \text{in } (0,\infty)\times\mathbb{R},\\
u(x,0) &= u_0(x), && \text{on } \mathbb{R}.
\end{aligned}
$$

Find general solutions via methods of characteristics:

- Ansatz characteristics: $\gamma(t) = (x(t), t)$ with $x'(t) = u(\gamma(t))$
- Solution on $\gamma$: $\nu(t) = u(\gamma(t))$ with $\nu'(t) \equiv 0$
- Hence, $x' = u(\gamma) = \nu \equiv c_1$, i.e. $x(t) = c_1 t + c_2$ is affine linear
- Thus, $u$ is **constant** on charac. which are **straight lines**

## Problem

Depending on the initial value, the solutions may not be unique.

- IV give $x(0) = c_2 = x(t) - c_1 t$ and
  $c_1 = \nu(t) = \nu(0) = u_0(x(0)) = u_0(c_2)$
- Thus,

$$u(x,t) = u_0(x - u(x,t)t)$$

# Burger's Equation

$$\begin{aligned} u_t + u\, u_x &\equiv 0, && \text{in } (0,\infty)\times\mathbb{R}, \\ u(x,0) &= u_0(x), && \text{on } \mathbb{R}. \end{aligned}$$

- Thus, $u$ is **constant** on charac. which are **straight lines**
- IV give $c_2 = x(0) = x(t) - c_1 t$ and $c_1 = \nu(t) = \nu(0) = u_0(x(0)) = u_0(c_2)$
- Thus,

$$u(x,t) = u_0(x - u(x,t)t)$$

For further analysis rewrite

$$x(t) = u_0(x(0))t + x(0)$$

# Burger's Equation

$$\begin{aligned} u_t + u\, u_x &\equiv 0, && \text{in } (0,\infty)\times\mathbb{R}, \\ u(x,0) &= u_0(x), && \text{on } \mathbb{R}. \end{aligned}$$

- Thus, $u$ is **constant** on charac. which are **straight lines**
- $x(t) = u_0(x(0))t + x(0)$
- $u(x,t) = u_0(x - u(x,t)t)$

Example

$$u(x,0) = \begin{cases} -2 & x < -2 \\ x & -2 \le x \le 2 \\ 2 & x \ge 2 \end{cases}$$

$$u(x,t) = \begin{cases} -2 & x \le -2(1+t) \\ \frac{x}{1+t} & -2(1+t) \le x \le 2(1+t) \\ 2 & x \ge 2(1+t) \end{cases}$$

# Burger's Equation

$$\begin{aligned} u_t + u\, u_x &\equiv 0, && \text{in } (0,\infty)\times\mathbb{R}, \\ u(x,0) &= u_0(x), && \text{on } \mathbb{R}. \end{aligned}$$

$$u(x,t) = u_0(x - u(x,t)t)$$

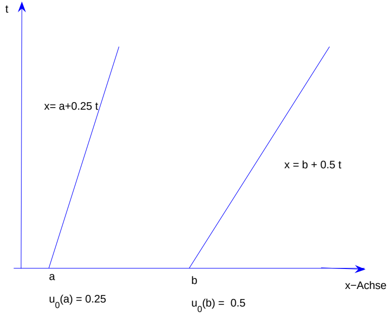
Four different cases

1. $u_0$ continuous and monotonically increasing: Uniqueness
2. $u_0$ continuous and not monotonically increasing: Non-uniqueness
3. $u_0$ monotonically increasing and not continuous: Region which is not touched by characteristic: rarefaction wave
4. $u_0$ jumps down: Characteristics intersect: Shock wave $s(t)$

# Burger's Equation

$$u_t + u\, u_x \equiv 0, \qquad \text{in } (0,\infty)\times\mathbb{R},$$
$$u(x,0) = u_0(x), \quad \text{on } \mathbb{R}.$$

- $u$ is **constant** on charac. which are **straight lines**
- $x(t) = u_0(x(0))t + x(0)$
- $u(x,t) = u_0(x - u(x,t)t)$




## Case 1: $u_0$ continuous and monotonically increasing: Uniqueness

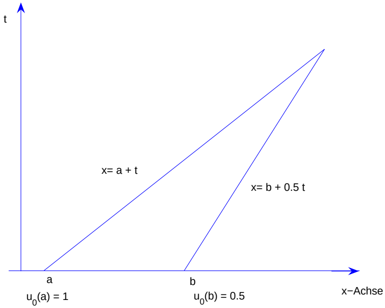
For $a < b$ consider characteristics $\gamma_a = (x_a(t), t)$ and $\gamma_b = (x_b(t), t)$ with $\gamma_a(0) = a$ and $\gamma_b(0) = b$ such that

$$x_a'(t) = u_0(a) < u_0(b) = x_b'(t)$$

# Burger's Equation

$$u_t + u\, u_x \equiv 0, \quad \text{in } (0,\infty) \times \mathbb{R},$$
$$u(x,0) = u_0(x), \quad \text{on } \mathbb{R}.$$

- $u$ is **constant** on charac. which are **straight lines**
- $x(t) = u_0(x(0))t + x(0)$
- $u(x,t) = u_0(x - u(x,t)t)$




## Case 2: $u_0$ continuous and not monotonically increasing: Non-uniqueness

For $a < b$ consider characteristics $\gamma_a = (x_a(t), t)$ and $\gamma_b = (x_b(t), t)$ with $\gamma_a(0) = a$ and $\gamma_b(0) = b$ such that

$$x_a'(t) = u_0(a) > u_0(b) = x_b'(t)$$

# Burger's Equation

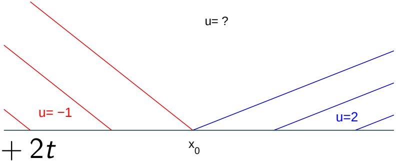
Case 3: $u_0$ monotonically increasing and not continuous: rarefaction wave

Region which is not touched by characteristics. The area without characteristics is filled by the rarefaction wave $u(x,t) = \frac{x-x_0}{t}$.

Example

$$u(x,0) = \begin{cases} -1 & x < x_0 \\ 2 & x \geq x_0 \end{cases}$$

$$u(x,t) = \begin{cases} -1 & x \leq x_0 - t, \\ ? & x_0 - t < x < x_0 + 2t \\ 2 & x \geq x_0 + 2t\,. \end{cases}$$



# Burger's Equation

## Case 3: $u_0$ monotonically increasing and not continuous: rarefaction wave

Region which is not touched by characteristics. The area without characteristics is filled by the rarefaction wave $u(x,t) = \frac{x - x_0}{t}$.

## Example (In General)

For $u_\ell < u_r$ and $u(x,0) = \begin{cases} u_\ell & x \leq x_0 \\ u_r & x > x_0 \end{cases}$ the solution to Burger's problem is

$$u(x,t) = \begin{cases} u_\ell & x \leq x_0 + u_\ell \cdot t, \\ \dfrac{x - x_0}{t} & x_0 + u_\ell \cdot t < x < x_0 + u_r \cdot t, \qquad t > 0 \\ u_r & x \geq x_0 + u_r \cdot t\,. \end{cases}$$

### Case 4: $u_0$ jumps down: Shock wave $s(t)$

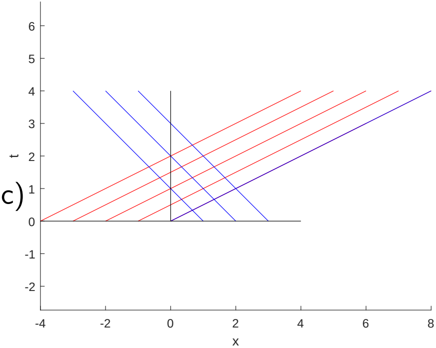
The characteristics intersect. The shock wave describes the movement of the discontinuity area.

### Example

$$u(x,0) = \begin{cases} u_\ell = 2 & x \leq x_0 \\ u_r = -1 & x > x_0 \end{cases}$$

### Questions

- How can functions with (drastic) discontinuities be solutions to PDEs?
- What is $s(t)$?



# Weak solutions

### Definition (Weak solution)

Consider $F \in \mathrm{C}^0(\mathbb{R})$ and $u \in \mathrm{L}^\infty_{\mathrm{loc}}(\mathbb{R})$. A **weak solution** resp. **integral solution** to the Cauchy problem for a conservation law

$$u_t + (F(u))_x \equiv 0 \qquad \text{in } (0,\infty) \times \mathbb{R}$$

is a function $u \in \mathrm{L}^\infty_{\mathrm{loc}}((0,\infty) \times \mathbb{R})$ such that

$$\int_0^\infty \int_{-\infty}^\infty (u\, v_t + F(u) v_x) dx dt + \int_{-\infty}^\infty u_0(x) v_0(x) dx = 0$$

holds for all test functions $v \in \mathrm{C}^1_{\mathrm{cpt}}([0,\infty) \times \mathbb{R})$.

# Weak solutions

## Rankie-Hugoniot Jump Condition

In case 4 where we expect a weak solution being a shock wave $s(t)$, it must hold

$$s'(t) = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r},$$

where $u_\ell$ and $u_r$ are the left and right initial values. In the special case of Burger's equations, this turns into (binomic formula)

$$s'(t) = \frac{\frac{u_\ell^2}{2} - \frac{u_r^2}{2}}{u_\ell - u_r} = \frac{u_\ell + u_r}{2}.$$

# Physically reasonable weak solutions to conservation law PDEs

We consider $F$ to be regular and strictly convex.

$$u_\ell + (F(u))_x = 0, \qquad \text{on } (0,\infty)\times\mathbb{R},$$

$$u(x,0) = \begin{cases} u_\ell, & x \le x_0 \\ u_r, & x > x_0 \end{cases}, \qquad \text{on } \mathbb{R}.$$

- $u_\ell > u_r$ : Shock wave $s(t)$ with $s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r}$, then
$$u(x,t) = \begin{cases} u_\ell & x \le s(t), \\ u_r & x > s(t). \end{cases}$$
- $u_\ell < u_r$: rarefaction wave, then
$$u(x,t) = \begin{cases} u_\ell & x \le x_0 + F'(u_\ell)t, \\ g\left(\dfrac{x-x_0}{t}\right) & x_0 + F'(u_\ell)t < x < x_0 + F'(u_r)t \\ u_r & x \ge x_0 + F'(u_r)t \end{cases}$$
with $g = (F')^{-1}$.

# Burger's equation

$u_\ell > u_r$ : Shock wave $s(t)$ with $s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r}$, then

$$u(x,t) = \begin{cases} u_\ell & x \leq s(t), \\ u_r & x > s(t)\,. \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = 2 & x < x_0 \\ u_r = -1 & x \geq x_0 \end{cases} \Rightarrow s'(t) =$$

$$u(x,t) = \begin{cases} u_\ell = 2 & x < \\ u_r = -1 & x \geq \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = -1 & x < x_0 \\ u_r = 2 & x \geq x_0 \end{cases} \Longrightarrow$$

$$u(x,t) = \begin{cases} u_\ell = -1 & x < \\ & \qquad \leq x \leq \\ u_r = 2 & x > \end{cases}$$

# Burger's equation

$u_\ell > u_r$ : Shock wave $s(t)$ with $s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r}$, $g = (F'^{-1}$ then

$$u(x,t) = \begin{cases} u_\ell & x \leq s(t), \\ u_r & x > s(t)\,. \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = 2 & x < x_0 \\ u_r = -1 & x \geq x_0 \end{cases} \Rightarrow s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r} = \frac{2+(-1)}{2} = \frac{1}{2}$$

$$u(x,t) = \begin{cases} u_\ell = 2 & x < x_0 + \frac{t}{2}, \\ u_r = -1 & x \geq x_0 + \frac{t}{2}. \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = -1 & x < x_0 \\ u_r = 2 & x \geq x_0 \end{cases} \Longrightarrow$$

$$u(x,t) = \begin{cases} u_\ell = -1 & x < \\ & \leq x \leq \\ u_r = 2 & x > \end{cases}$$

# Burger's equation

$u_\ell < u_r$: rarefaction wave, then

$$u(x,t) = \begin{cases} u_\ell & x \le x_0 + F'(u_\ell)t, \\ g\left(\dfrac{x-x_0}{t}\right) & x_0 + F'(u_\ell)t < x < x_0 + F'(u_r)t \\ u_r & x \ge x_0 + F'(u_r)t \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = 2 & x < x_0 \\ u_r = -1 & x \ge x_0 \end{cases} \Rightarrow s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r} = \frac{2+(-1)}{2} = \frac{1}{2}$$

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$$u(x,t) = \begin{cases} u_\ell = -1 & x < \\ & \le x \le \\ u_r = 2 & x > \end{cases}$$

# Burger's equation

$u_\ell < u_r$: rarefaction wave, then

$$u(x,t) = \begin{cases} u_\ell & x \leq x_0 + F'(u_\ell)t, \\ g\left(\dfrac{x-x_0}{t}\right) & x_0 + F'(u_\ell)t < x < x_0 + F'(u_r)t \\ u_r & x \geq x_0 + F'(u_r)t \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = 2 & x < x_0 \\ u_r = -1 & x \geq x_0 \end{cases} \Rightarrow s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r} = \frac{2+(-1)}{2} = \frac{1}{2}$$

$$u(x,t) = \begin{cases} u_\ell = 2 & x < x_0 + \frac{t}{2}, \\ u_r = -1 & x \geq x_0 + \frac{t}{2}. \end{cases}$$

$$u(x,0) = \begin{cases} u_\ell = -1 & x < x_0 \\ u_r = 2 & x \geq x_0 \end{cases} \Longrightarrow g(x) = (F')^{-1}(x) = x$$

$$u(x,t) = \begin{cases} u_\ell = -1 & x < x_0 - t, \\ \frac{x-x_0}{t} & x_0 - t \leq x \leq x_0 + 2t \\ u_r = 2 & x > x_0 + 2t. \end{cases}$$

# Burger's Equation Example

$u_\ell < u_r$: Rarefaction wave, then

$$u(x,t) = \begin{cases} u_\ell & x \leq x_0 + F'(u_\ell)t, \\ g\left(\frac{x-x_0}{t}\right) & x_0 + F'(u_\ell)t < x < x_0 + F'(u_r)t \text{ with} \\ u_r & x \geq x_0 + F'(u_r)t \end{cases}$$

$g = (F')^{-1}$.

$$u(x,0) = \begin{cases} -1 & x \leq -2\,, \\ 0 & -2 < x \leq 3\,, \\ 1 & x > 3\,. \end{cases}$$

$$u(x,t) = \begin{cases} -1 & x \leq \\ & < x \leq \\ 0 & < x \leq \\ & < x \leq \\ 1 & x > \end{cases}$$

# Burger's Equation Example

$u_\ell < u_r$: rarefaction wave, then

$$u(x,t) = \begin{cases} u_\ell & x \le x_0 + F'(u_\ell)t, \\ g\left(\dfrac{x - x_0}{t}\right) & x_0 + F'(u_\ell)t < x < x_0 + F'(u_r)t \\ u_r & x \ge x_0 + F'(u_r)t \end{cases} \text{ with}$$

$g = (F')^{-1}$.

$$u(x,0) = \begin{cases} -1 & x \le -2\,, \\ 0 & -2 < x \le 3\,, \\ 1 & x > 3\,. \end{cases}$$

$$u(x,t) = \begin{cases} -1 & x \le -2 - t \\ \frac{x+2}{t} & -2 - t < x \le -2 \\ 0 & -2 < x \le 3 \\ \frac{x-3}{t} & 3 < x \le 3 + t \\ 1 & x > 3 + t \end{cases}$$

# Burger's Equation Example

$u_\ell > u_r$ : Shock wave $s(t)$ with $s'(t) = \frac{F(u_\ell)-F(u_r)}{u_\ell - u_r}$, then

$$u(x,t) = \begin{cases} u_\ell & x \leq s(t), \\ u_r & x > s(t)\,. \end{cases}$$

$$u(x,0) = \begin{cases} \frac{1}{2} & x \leq -1\,, \\ -1 & -1 < x \leq 1\,, \\ -2 & x > 1\,. \end{cases}$$

First we get

$s_1(t) =$

$s_2(t) =$

$$u(x,t) = \begin{cases} \frac{1}{2} & x \leq \\ -1 & \quad < x \leq \\ -2 & x > \end{cases}$$

# Burger's Equation Example

$u_\ell > u_r$ : Shock wave $s(t)$ with $s'(t) = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r}$, then

$$u(x,t) = \begin{cases} u_\ell & x \le s(t), \\ u_r & x > s(t). \end{cases}$$

$$u(x,0) = \begin{cases} \frac{1}{2} & x \le -1, \\ -1 & -1 < x \le 1, \\ -2 & x > 1. \end{cases}$$

First we get

$s_1(t) = -\frac{1}{4}t - 1$

$s_2(t) = -\frac{3}{2}t + 1$

$$u(x,t) = \begin{cases} \frac{1}{2} & x \le -\frac{1}{4}t - 1 \\ -1 & -\frac{1}{4}t - 1 < x \le -\frac{3}{2}t + 1 \\ -2 & x > -\frac{3}{2}t + 1 \end{cases}$$

# Burger's Equation Example

$u(x,0) = \begin{cases} \frac{1}{2} & x \le -1\,, \\ -1 & -1 < x \le 1\,, \\ -2 & x > 1\,. \end{cases}$ First we get $s_1(t) = -\frac{1}{4}t - 1$,

$s_2(t) = -\frac{3}{2}t + 1$, thus, $u(x,t) = \begin{cases} \frac{1}{2} & x \le -\frac{1}{4}t - 1 \\ -1 & -\frac{1}{4}t - 1 < x \le -\frac{3}{2}t + 1 \\ -2 & x > -\frac{3}{2}t + 1 \end{cases}$

- This only works out until the shock waves meet:

$$s_1(t^*) = s_2(t^*) \Leftrightarrow t^* = \frac{8}{5}$$

- New shock wave with $u_\ell = \frac{1}{2}$ and $u_r = -2$

$$s_3(t) = s_1(t^*) + s_3'(t)(t - t^*) \quad \text{with } s_3' = \frac{u_\ell + u_r}{2} = -\frac{3}{4}$$

- Hence,

$$u(x,t) = \begin{cases} \frac{1}{2} & x \le -\frac{3}{4}t - \frac{1}{5}, \\ -2 & x > -\frac{3}{4}t - \frac{1}{5} \end{cases} \quad \text{for } t > t^*.$$

# Burger's Equation Example

$$u(x,0) = \begin{cases} -\frac{1}{2} & x \le 0\,, \\ 0 & 0 < x \le \frac{1}{2}\,, \\ -1 & x > \frac{1}{2}\,. \end{cases}$$ For small $t$, we can continue as usual:

$$u(x,t) = \begin{cases} -\frac{1}{2} & x \le -\frac{1}{2}t \\ \frac{x-0}{t} & -\frac{1}{2}t < x \le 0 \\ 0 & 0 < x \le -\frac{1}{2}t + \frac{1}{2} \\ -1 & x > -\frac{1}{2}t + \frac{1}{2} \end{cases}$$

This can only be true for $t^* \le 1$. Find new shock wave $s(t)$ with

- $u_\ell = \frac{x}{t} = \frac{s(t)}{t}$, $u_r = -1 \implies s'(t) = \dfrac{\frac{s(t)}{t} - 1}{2}$
- $s'(t) = \frac{1}{2t} \cdot s(t) - \frac{1}{2}$ (linear ODE to solve for instance with homogenization and variation of constant)
- Solution: $s(t) = c\sqrt{t} - t$
- Since $s(1) = 0$, get $c = 1$

# Burger's Equation Example

Hence, for $t > 1$, we found

$$u(x,t) = \begin{cases} -\frac{1}{2} & x \le -\frac{t}{2} \\ \frac{x}{t} & -\frac{t}{2} < x \le \sqrt{t} - t \\ -1 & x > \sqrt{t} - t \end{cases}$$

which by itself is only valid for $t \le 4$. At this time, the rarefaction wave is faded. The wave front is then described by

$$\tilde{s}(t) = s(4) + \tilde{s}'(t)(t-4) = -2 + \frac{-\frac{1}{2} - 1}{2} \cdot (t-4),$$

which gives

$$u(x,t) = \begin{cases} -\frac{1}{2} & x \le -\frac{3}{4}\, t + 1\,, \\ -1 & x > -\frac{3}{4}\, t + 1\,. \end{cases}$$

# Another Conservation Law

$$u_t + \left((u+1)^2\right)_x \;=\; 0\,, \qquad u(x,0) \;=\; u_0(x)$$

- $F(y) = (y+1)^2$
- $F'(y) = 2(y+1)$
- $g(y) = (F')^{-1}(y) = \frac{y}{2} - 1$
- $g(\frac{x-x_0}{t}) = \frac{x-x_0}{2t} - 1$

For example for $u(x,0) = \begin{cases} -1 & x \le 3 \\ 0 & x > 3 \end{cases}$ we derive the rarefaction wave

$$u(x,t) = \begin{cases} -1 & x \le 3 + -\frac{1}{2}t, \\ \dfrac{x-3}{t} - 1 & 3 + -\frac{1}{2}t < x < 3 - t \\ 0 & x \ge 3 - t\,. \end{cases}$$

# Another Conservation Law

For $u_\ell > u_r$ there must be a rarefaction wave inserted

$$\dot{s}(t) = \frac{F(u_\ell) - F(u_r)}{u_\ell - u_r} = \frac{(u_\ell + 1)^2 - (u_r + 1)^2}{u_\ell - u_r} = 1.$$

For example

$$u(x,0) = \begin{cases} 0 & x \leq 3 \\ -1 & x > 3 \end{cases}$$

$$u(x,t) = \begin{cases} 0 & x \leq s(t) = 3 + t \\ -1 & x > s(t) = 3 + t. \end{cases}$$

# Concerning Homework 3

- As a reminder: The conservation equation is

$$(\text{ density })_t + (\text{ flow })_x = 0 \qquad \text{here: } u_t + q_x = 0,$$

  where flow = velocity · density
- If you do not manage part a), continue in b) with

$$u_t + \left(u \cdot v_{max}\,(1 - \frac{u}{u_{max}})\right)_x = 0$$

- The condition for spurious waves is $u_\ell > u_r$ for Burgers, but in general:

$$F'(u_\ell) > F'(u_r) \qquad \text{here} \qquad q'(u_\ell) > q'(u_r)$$

- The jump condition must apply!
- Interference waves: Characteristics run into interference front!