Auditorium Exercise 02

Quasilinear first-order PDE
Method of characteristics

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Announcements

Exercise class Monday, May 5th, 11:30

Overview

Method of Characteristics: Homogeneous PDEs

Examples

Continuity Equation: Reminder

- $u(x,t)$: density
- $q(x,t)$: flow
- $M(t) = \int_\Omega u(x,t)dx$ mass
- also

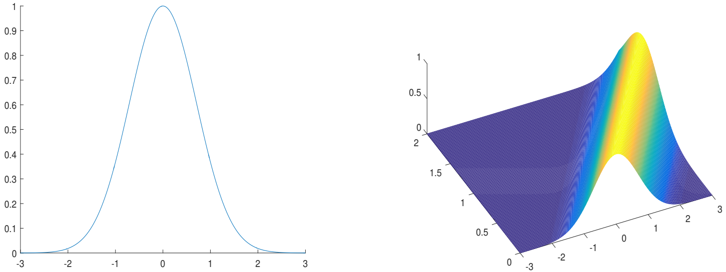
$$M(t) = M(t_0) + \int_{t_0}^{t} \left(\int_{\partial\Omega} q(x,t) \cdot n(x,t) dx \right) dt$$

  comp. HW 2
- By some computation and regularity assumptions, we got

$$u_t(x,t) + \nabla q(x,t) = 0$$

Continuity Equation

- Scalar case: $u_t + (q(u,v))_x = 0$ ($u = u(x,t)$, $v = v(x,t)$)
- Easiest case: $q(u,v) = const \cdot u$: transport equation
- density profile moves without change along the time variable
  $u_t + cu_x = 0$



(a) Start function $u(x,0)$ (b) Solution $u(x,t)$

Figure: Start function and corresponding solution

Method of characteristics for the **homogeneous** continuity equation with initial values $u(x, t_0) = u_0$

Basic idea to solve these equations:

1. Find curves $(x(t), t)$ in the (x, t)-plane along which u is **constant**. (=characteristics)
2. Find intersection of this curve with the initial values
3. Interpreting the results in the right way gives the solution

Strategy: Step 1

Step 1: On the **characteristic curves**, we want $u(x(t), t) = K$, so:

$\frac{d}{dt} u(x(t), t) =$

Strategy: Step 2

Step 2: For the transport equation $u_t + cu_x = 0$ with initial condition $u(x, 0) = f(x)$

$\dot{x}(t) =$

$x(t) =$

$x(0) =$

$u(x(t), t) = u(x(0), 0)$

Now a bit more generally:

Case A: Initial Value Problem for a Linear Homogeneous PDE

Coefficients do not depend on u.

$$\beta(x,t)u_t(x,t) + a(x,t)u_x(x,t) = 0, \quad x \in \mathbb{R},\ t > 0$$
$$u(x,0) = g(x), \quad x \in \mathbb{R}$$

Case A - Characteristics

Coefficients do not depend on u.

$$\begin{aligned} &\beta(x,t)u_t(x,t) + a(x,t)u_x(x,t) = 0, \quad x \in \mathbb{R},\ t > 0 \\ &u(x,0) = g(x), \quad x \in \mathbb{R} \end{aligned} \tag{1}$$

Step 1: Determine characteristic curves $s \mapsto (x(s), t(s))^T$ with:

$$\frac{dt}{ds} = \beta(x,t), \quad \frac{dx}{ds} = a(x,t)$$

Then along these **characteristic curves**:

$$\frac{d}{ds}\, u\,(x(s), t(s)) \;=\; u_x \cdot \frac{dx}{ds} \;+\; u_t \cdot \frac{dt}{ds} \;=\; \beta\, u_t \;+\; a\, u_x\,.$$

Case A - Solution

Thus, for every solution u of PDE $\dfrac{d}{ds}\,u\,(x(s), t(s))\, = 0$.

Solution of PDE (1) u is (in homogeneous case) constant along characteristic curves

Case A - Alternative Soltuion

Alternatively, a simpler and faster concrete solution is available in the case $\beta(x,t) \neq 0$, use t as the parameter.

We had $\dfrac{dt}{ds} = \beta(x,t) \qquad \dfrac{dx}{ds} = a(x,t)$

The characteristic curves are $t \mapsto \binom{x(t)}{t}$ with

$$\dot{x}(t) = \frac{dx}{dt} = \frac{a(x,t)}{\beta(x,t)}\,.$$

Step 2: Find where the characteristic through (x,t) intersects the initial condition curve.
Step 3: Read $u(x,t)$ from initial values or solve for it explicitly.

Example 1: Another Transport Equation

$$2u_t - 4u_x = 0, \quad u(x,0) = \sin\left(\frac{x}{2}\right)$$

$$\frac{dt}{ds} = \qquad , \frac{dx}{ds} = \qquad \text{or} \quad \frac{dx}{dt} =$$

$$\implies x(t) =$$

Example 1: Transport Equation

$2u_t - 4u_x = 0, \quad u(x,0) = \sin\left(\frac{x}{2}\right)$

$\frac{dt}{ds} =$, $\frac{dx}{ds} =$ or $\frac{dx}{dt} =$

$\Longrightarrow x(t) =$

Solution constant along the lines $\begin{pmatrix} x(t) \\ t \end{pmatrix}$ with $x =$

Intersection of the line with thex-axis
(Initial values): $x(0) = k$
On the characteristic curves: $u(x(t),t) = u(x(0),0) = \sin\left(\frac{x(0)}{2}\right)$
How does k=x(0) depend on x and t?

k=

Solution: u(x,t)=

Be aware

Initial values cannot always be prescribed arbitrarily. Some choices lead to no solution or non-uniqueness.

The PDE above with

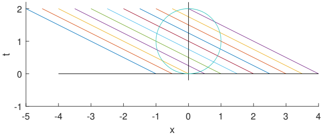
$$u(-2a, a) = g(a)$$

$\Longrightarrow$ no solution or infinite many solutions

The PDE above with

$$u(x, t) = x \quad for \quad \sqrt{(t-1)^2 + x^2} = 1$$

$\Longrightarrow$ no solution



Example 2

$$\begin{aligned} 1 \cdot u_t + (x+1)u_x &= 0 \qquad x \in \mathbb{R},\ t > 0, \\ u(x,0) &= g(x) \qquad x \in \mathbb{R} \end{aligned}$$

Then the following holds for the characteristic curves.

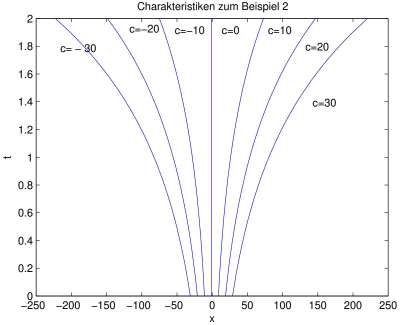
$$\frac{dt}{ds} = 1\,, \qquad \frac{dx}{ds} = x + 1\,, \qquad \text{or} \qquad \frac{dx}{dt} = x + 1$$

For $x \neq -1$:

$$\frac{dx}{x+1} = dt \implies \ln|x+1| = k + t \implies x(t) = ce^t - 1$$

Example 2

PDE: $u_t + (x+1)u_x = 0$ states for $x = -1$:
$u_t = 0 \implies u(-1,t) = u(-1,0) = g(-1)$.



Along the curves, the solution is constant and depends only on c.

Example 2

We had $x(t) = ce^t - 1$ so:

$$c =$$

For $t = 0$ holds $\qquad x_0 := x(0) = c - 1$

The (x, t) associated with x_0 is thus $\boxed{x_0 = \qquad\qquad\qquad\qquad}$

Along each characteristic curve, it again holds that

$$u(x,t) = u(x_0, 0) = g(x_0) = g\left(\qquad\qquad\qquad\qquad \right)$$

Example 3: One dimension more

$2xu_x + yu_y + tu_t = 0, \qquad x, y \in \mathbb{R},\ t \in \mathbb{R}^+$

E.g. equipped with the initial condition $u(x, y, 2) = \sin(x)e^{-y}$.

Find curves $(x(s), y(s), t(s))$ such that $u(x(s), y(s), t(s))$ is constant!

Then it holds: $\frac{d}{ds}u(x(s), y(s), t(s)) =$

Example 3: One dimension more

Characteristic differential equation system:

Example 3: One dimension more

Or with t as parameter for $t \neq 0$ for $2xu_x + yu_y + tu_t = 0$

General Case: Quasi-linear First Order PDEs

Coefficients may depend on u; may be inhomogeneous.

Example:

$$1 * u_t(x,t) + a(x,t,u)u_x(x,t) = b(x,t,u)$$

Auxiliary Problem: Find a function $U(x,t,u)$ satisfying:

$$1 \cdot U_t \; + \; a(x,t,u) \cdot U_x \; + \; b(x,t,u) \cdot U_u \; = \; 0 \qquad (2)$$

Characteristic system of differential equations with s as parameter:

$$\frac{dt}{ds} = 1, \quad \frac{dx}{ds} = a(x,t,u), \quad \frac{du}{ds} = b(x,t,u)$$

Or with t as parameter:

$$\frac{dx}{dt} \; = \; a(x,t,u)\,, \qquad \frac{du}{dt} \; = \; b(x,t,u). \qquad (3)$$

General Case: Quasi-linear First Order PDEs

Then it holds for every solution U der PDE (2) along the characteristic curves:

$$\frac{d}{ds}\,U\,(x(s),t(s),u(s)) \;=\; U_x\cdot\frac{dx}{ds} \;+\; U_t\cdot\frac{dt}{ds} \;+\; U_u\cdot\frac{du}{ds}$$
$$=\; U_t \;+\; a\,U_x \;+\; b\,U_u =\; 0$$

Integrate the system (3) to obtain C_1, C_2 constants.

Characteristic curves are defined by these parameters:
$U(x,t,u) = \tilde{\Phi}(C_1,C_2) = K$ constant on characteristic curves

$$\Phi = \tilde{\Phi} - K \implies \Phi(C_1,C_2) = \Phi(C_1(x,t,u),C_2(x,t,u)) = 0$$

One equation for 2 unknows: solve if possible $C_2 = f(C_1)$

Interlude

Justification for using the auxiliary PDE (2) for the original problem.
On every characteristic curve $U(x, t, u) - K = 0$.
If $U_u \neq 0$, by the **implicit function theorem**,

$$u = u(x, t), \quad \begin{pmatrix} u_x \\ u_t \end{pmatrix} = -U_u^{-1} \begin{pmatrix} U_x \\ U_t \end{pmatrix}$$

Then: $U_x = -u_x \cdot U_u \qquad U_t = -u_t \cdot U_u$

$$\begin{aligned} U_t + a\, U_x + b\, U_u = 0 &\iff -u_t \cdot U_u - a\, u_x \cdot U_u + b\, U_u = 0 \\ &\iff -u_t - a\, u_x + b = 0 \end{aligned}$$

u solves the original PDE

Example 1:Linear inhomogeneous initial value problem

$$u_t - 2u_x = t, \qquad u(x,0) = \frac{1}{1+x^2}$$

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Step 1: Set up the characteristic system of differential equations:

$\frac{dx}{dt} =$ $\frac{du}{dt} =$

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Step 1: Set up the characteristic system of differential equations:

$\frac{dx}{dt} =$ $\qquad$ $\frac{du}{dt} =$

Step 2: Solve and express the constants of integration using the variables.

$\Longrightarrow x(t) =$ $\qquad$ $u(t) =$

$C =$ $\qquad$ $D =$

Example 1:Linear inhomogeneous initial value problem

$$u_t - 2u_x = t, \qquad u(x,0) = \frac{1}{1+x^2}$$

Step 1: Set up the characteristic system of differential equations:

$\frac{dx}{dt} =$ $\qquad\qquad$ $\frac{du}{dt} =$

Step 2: Solve and express the constants of integration using the variables.

$\implies x(t) =$ $\qquad\qquad$ $u(t) =$

$C =$ $\qquad\qquad$ $D =$

Example 1:Linear inhomogeneous initial value problem

Step 3: If possible, solve $D = f(C)$ for u.

Step 4: Determine f and thus u using the initial values.

Example 2

$$u_x + y^2 u_y = u^2, \qquad u(x,1) = 1 \qquad x \in \mathbb{R},\ y \in \mathbb{R}^+$$

Auxiliary problem: $\qquad U$

Example 2

$$u_x + y^2 u_y = u^2, \qquad u(x,1) = 1 \qquad x \in \mathbb{R},\ y \in \mathbb{R}^+$$

Auxiliary problem: $\quad U_x + y^2 U_y + u^2 \cdot U_u = 0$

Example 2: Step 1

$$u_x + y^2 u_y = u^2, \qquad u(x,1) = 1 \qquad x \in \mathbb{R},\ y \in \mathbb{R}^+$$

Set up the characteristic system of differential equations:

$$\frac{dy}{dx} = \qquad , \qquad \frac{du}{dx} =$$

Example 2: Step 2

Solve, express integration constants using the variables

$$\int \frac{dy}{y^2} = \int dx \,, \Longrightarrow -\frac{1}{y} = x - C_1 \Longrightarrow \boxed{C_1 = \frac{1}{y} + x}$$

Completely analogously, we obtain $\boxed{C_2 = \frac{1}{u} + x}$

Example 2: Step 3

If possible, solve $D = f(C)$ for u.

Solution satisfies: $\Phi(C_1, C_2) = \Phi(\frac{1}{y} + x, \frac{1}{u} + x) = 0$

In the case of solvability, the following holds: $C_2 = f(C_1)$

$$\frac{1}{u} + x =$$

Example 2: Step 3

If possible, solve $D = f(C)$ for u.

Solution satisfies: $\Phi(C_1, C_2) = \Phi(\frac{1}{y} + x, \frac{1}{u} + x) = 0$

In the case of solvability, the following holds: $C_2 = f(C_1)$

$$\frac{1}{u} + x = f(\frac{1}{y} + x) \implies \frac{1}{u} = f(\frac{1}{y} + x) - x$$

Example 2: Step 4

Determine f and thus u using the initial values.

Initial condition $\quad u(x,1) \stackrel{!}{=} 1$:

$$u(x,y) = \frac{1}{f\left(\frac{1}{y} + x\right) - x} \xrightarrow{y=1}$$

Example 3:

Determine the solution of the initial value problem.

$$\begin{aligned} u_t + 2xu_x &= tu, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(x,0) &= \sin(x). \end{aligned}$$

Example 3

Solution: For the characteristics, the following holds

$$\frac{dx}{dt} = 2x \quad \Longrightarrow$$

$$x(t) = c_1 e^{2t}, \qquad c_1 = xe^{-2t}.$$

On the characteristic curves, the following holds

$$\frac{du}{dt} = tu \quad \Longrightarrow$$

$$u(x(t), t) = c_2 e^{\frac{t^2}{2}} \Longrightarrow$$

Example 3

So $c_2 = \; ue^{-\frac{t^2}{2}}$
With a suitable Φ then holds: $\Phi(c_1, c_2) = 0 \implies$

If solvable according to c_2 : $c_2 = \; f(c_1)$:
$c_2 =$

On the other hand $u(x, 0) \overset{!}{=} \sin(x)$, therefore

$u(x, 0) =$

Thus, we obtain the solution.

$$u(x, t) = \sin(xe^{-2t}) \cdot e^{\frac{t^2}{2}} .$$

Example 4: ARWA

Determine the solution $u(x,t)$ of the following initial-boundary value problem.

$$\begin{aligned} u_t + 2u_x + u &= 0, \qquad x,t > 0 \\ u(x,0) &= 0 \qquad (x \geq 0) \\ u(0,t) &= t^2 \qquad (t \geq 0) \end{aligned}$$

by means of the method of characteristics.

Example 4: ARWA

Idea: Determine the solution $u(x,t)$ of the following initial-boundary value problem.

$$\begin{aligned} u_t + 2u_x + u &= 0, \qquad x, t > 0 \\ u(x,0) &= 0 \qquad (x \geq 0) \\ u(0,t) &= t^2 \qquad (t \geq 0) \end{aligned}$$

by means of the method of characteristics.

Idea: Determine a solution for each initial condition $u(x,0) = 0$, and one for the boundary condition $u(0,t) = t^2$ and smoothly assemble these solutions.

Example 4: ARWA

Using the method of characteristics, we obtain: $\dfrac{dx}{dt} = 2, \; \dfrac{du}{dt} = -u$

$x(t) = 2t + C_1 \qquad u(x(t), t) = C_2 e^{-t}$

$\implies C_1 = x - 2t, \; C_2 = ue^t$

$\Phi\left(x - 2t, ue^t\right) = 0$ yields a solution if solvable

$u = e^{-t} f(x - 2t).$

Example 4: ARWA

From the boundary values $u(0,t) = t^2$ we obtain

$$t^2 = e^{-t} f(0 - 2t)$$

or with $\mu = -2t$ i.e. $t = -\frac{\mu}{2}$

$$f(\mu) = \left(-\tfrac{\mu}{2}\right)^2 e^{-\frac{\mu}{2}}$$

and thus one obtains for the solution

$$u_R(x,t) = e^{-t} f(x - 2t) = e^{-t} \left(-\frac{x-2t}{2}\right)^2 e^{\frac{-(x-2t)}{2}} =$$

$$e^{-\frac{x}{2}} \left(\frac{2t - x}{2}\right)^2$$

Example 4: ARWA

On the other hand, if one substitutes into the general solution
$u(x,t) = e^{-t}\tilde{f}(x-2t)$
Substituting the initial data $u(x,0) = 0$, we obtain:

The solutions can be smoothly combined along curves where $u = \tilde{u}$ holds. This is the case along the line $x = 2t$. Therefore, we obtain as the solution to the original problem:

$$u(x,t) = \begin{cases} e^{-\frac{x}{2}} \left(\frac{2t-x}{2}\right)^2 & x \leq 2t \\ 0 & x > 2t \end{cases}$$

It would still need to be verified whether the solution along the line $x = 2t$ is continuously (partially) differentiable.