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Fachbereich Mathematik der Universität Hamburg

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Hörsaalübung 2 Differentialgleichungen II Quasilineare Differentialgleichungen erster Ordnung Charakteristikenmethode

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Auditorium Exercise 02 Quasilinear first-order PDE Method of characteristics

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Announcements

Exercise class Monday, May 5th, 11:30



Method of Characteristics: Homogeneous PDEs

Examples

Continuity Equation: Reminder

•
$$u(x, t)$$
: density
• $q(x, t)$: flow
• $M(t) = \int_{\Omega} u(x, t) dx$ mass
• also

$$M(t) = M(t_0) + \int_{t_0}^t \left(\int_{\partial \Omega} q(x, t) \cdot n(x, t) dx \right) dt$$

comp. HW 2

By some computation and regularity assumptions, we got

$$u_t(x,t)+\nabla q(x,t)=0$$

Continuity Equation

- Scalar case: $u_t + (q(u, v))_x = 0$ (u = u(x, t), v = v(x, t))
- Easiest case: $q(u, v) = const \cdot u$: transport equation
- density profile moves without change along the time variable u_t + cu_x = 0



Figure: Start function and corresponding solution

Method of characteristics for the **homogeneous** continuity equation with initial values $u(x, t_0) = u_0$

Basic idea to solve these equations:

- Find curves (x(t), t) in the (x, t)-plane along which u is constant. (=characteristics)
- 2. Find intersection of this curve with the initial values
- 3. Interpreting the results in the right way gives the solution

Strategy: Step 1

Step 1: On the characteristic curves, we want u(x(t), t) = K, so:

 $\frac{d}{dt}u(x(t),t) =$

Strategy: Step 2

Step 2: For the transport equation $u_t + cu_x = 0$ with initial condition u(x, 0) = f(x)

 $\dot{x}(t) =$

x(t) =

x(0) =

u(x(t),t)=u(x(0),0)

Now a bit more generally:

Case A: Initial Value Problem for a Linear Homogeneous PDE

Coefficients do not depend on u.

$$egin{aligned} eta(x,t)u_t(x,t)+a(x,t)u_x(x,t)&=0,\quad x\in\mathbb{R},\ t>0\ u(x,0)&=g(x),\quad x\in\mathbb{R} \end{aligned}$$

Case A - Characteristics

Coefficients do not depend on u.

$$\begin{aligned} \beta(x,t)u_t(x,t) + a(x,t)u_x(x,t) &= 0, \quad x \in \mathbb{R}, \ t > 0 \\ u(x,0) &= g(x), \quad x \in \mathbb{R} \end{aligned}$$

Step 1: Determine characteristic curves $s \mapsto (x(s), t(s))^T$ with:

$$rac{dt}{ds} = eta(x,t), \quad rac{dx}{ds} = a(x,t)$$

Then along these characteristic curves:

$$\frac{d}{ds} u(x(s), t(s)) = u_x \cdot \frac{dx}{ds} + u_t \cdot \frac{dt}{ds} = \beta u_t + a u_x \cdot \frac{dt}{ds}$$

Case A - Solution

Thus, for every solution u of PDE $\frac{d}{ds}u(x(s), t(s)) = 0$.

Solution of PDE (1) u is (in homogeneous case) constant along characteristic curves

Case A - Alternative Soltuion

Alternatively, a simpler and faster concrete solution is available in the case $\beta(x, t) \neq 0$, use t as the parameter.

We had
$$\frac{dt}{ds} = \beta(x,t)$$
 $\frac{dx}{ds} = a(x,t)$

The characteristic curves are $t \mapsto \binom{x(t)}{t}$ with

$$\dot{x}(t) = rac{dx}{dt} = rac{a(x,t)}{eta(x,t)}.$$

Step 2: Find where the characteristic through (x, t) intersects the initial condition curve.

Step 3: Read u(x, t) from initial values or solve for it explicitly.

Example 1: Another Transport Equation

$$2u_t-4u_x=0, \quad u(x,0)=\sin\left(\frac{x}{2}\right)$$

$$\frac{dt}{ds} = , \frac{dx}{ds} =$$
 or $\frac{dx}{dt} =$

 $\implies x(t) =$

Example 1: Transport Equation

$$2u_t - 4u_x = 0, \quad u(x,0) = \sin\left(\frac{x}{2}\right)$$

$$\frac{dt}{ds} =$$
, $\frac{dx}{ds} =$ or $\frac{dx}{dt} =$

$$\implies x(t) =$$

Solution constant along the lines $\begin{pmatrix} x(t) \\ t \end{pmatrix}$ with $x =$

Intersection of the line with thex-axis (Initial values): x(0) = kOn the characteristic curves: $u(x(t), t) = u(x(0), 0) = \sin\left(\frac{x(0)}{2}\right)$ How does k=x(0) depend on x and t?

k =

Solution: u(x,t) =

Be aware

Initial values cannot always be prescribed arbitrarily. Some choices lead to no solution or non-uniqueness.

The PDE above with

$$u(-2a,a)=g(a)$$

 \Longrightarrow no solution or infinite many solutions

The PDE above with

$$u(x,t) = x$$
 for $\sqrt{(t-1)^2 + x^2} = 1$

 \implies no solution



$$egin{array}{ll} 1\cdot u_t+(x+1)u_x=0 & x\in \mathbb{R}, \ t>0, \ u(x,0)=g(x) & x\in \mathbb{R} \end{array}$$

Then the following holds for the characteristic curves.

$$\frac{dt}{ds} = 1, \qquad \frac{dx}{ds} = x+1, \qquad \text{or} \quad \frac{dx}{dt} = x+1$$
For $x \neq -1$:
$$\frac{dx}{x+1} = dt \implies \ln|x+1| = k+t \implies x(t) = ce^t - 1$$





Along the curves, the solution is constant and depends only on c.

We had $x(t) = ce^t - 1$ so:

c =

For t = 0 holds $x_0 := x(0) = c - 1$ The (x, t) associated with x_0 is thus $x_0 = 1$

$$x_0 = 1$$

Along each characteristic curve, it again holds that

$$u(x,t) = u(x_0,0) = g(x_0) = g($$
)

Example 3: One dimension more

 $2xu_x + yu_y + tu_t = 0, \qquad x, y \in \mathbb{R}, \ t \in \mathbb{R}^+$

E.g. equipped with the initial condition $u(x, y, 2) = \sin(x)e^{-y}$.

Find curves (x(s), y(s), t(s)) such that u(x(s), y(s), t(s)) is constant!

Then it holds: $\frac{d}{ds}u(x(s), y(s), t(s)) =$

Example 3: One dimension more

Characteristic differential equation system:

Example 3: One dimension more

Or with t as parameter for $t \neq 0$ for $2xu_x + yu_y + tu_t = 0$

General Case: Quasi-linear First Order PDEs

Coefficients may depend on u; may be inhomogeneous.

Example:

$$1 * u_t(x, t) + a(x, t, u)u_x(x, t) = b(x, t, u)$$

Auxiliary Problem: Find a function U(x, t, u) satisfying:

$$1 \cdot U_t + a(x,t,u) \cdot U_x + b(x,t,u) \cdot U_u = 0 \qquad (2)$$

Characteristic system of differential equations with *s* as parameter:

$$rac{dt}{ds} = 1, \quad rac{dx}{ds} = a(x, t, u), \quad rac{du}{ds} = b(x, t, u)$$

Or with *t* as parameter:

$$\frac{dx}{dt} = a(x, t, u), \qquad \frac{du}{dt} = b(x, t, u). \tag{3}$$

General Case: Quasi-linear First Order PDEs

Then it holds for every solution U der PDE (2) along the characteristic curves:

$$\frac{d}{ds} U(x(s), t(s), u(s)) = U_x \cdot \frac{dx}{ds} + U_t \cdot \frac{dt}{ds} + U_u \cdot \frac{du}{ds}$$
$$= U_t + a U_x + b U_u = 0$$

Integrate the system (3) to obtain C_1, C_2 constants.

Characteristic curves are defined by these parameters: $U(x, t, u) = \tilde{\Phi}(C_1, C_2) = K$ constant on characteristic curves

$$\Phi = \tilde{\Phi} - K \implies \Phi(C_1, C_2) = \Phi(C_1(x, t, u), C_2(x, t, u)) = 0$$

One equation for 2 unknows: solve if possible $C_2 = f(C_1)$

Interlude

Justification for using the auxiliary PDE (2) for the original problem.

On every characteristic curve U(x, t, u) - K = 0. If $U_u \neq 0$, by the **implicit function theorem**,

$$u = u(x, t), \quad \begin{pmatrix} u_x \\ u_t \end{pmatrix} = -U_u^{-1} \begin{pmatrix} U_x \\ U_t \end{pmatrix}$$

Then: $U_x = -u_x \cdot U_u$ $U_t = -u_t \cdot U_u$

 $U_t + a U_x + b U_u = 0 \iff -u_t \cdot U_u - a u_x \cdot U_u + b U_u = 0$ $\iff -u_t - a u_x + b = 0$

u solves the original PDE

$$u_t - 2u_x = t,$$
 $u(x,0) = \frac{1}{1+x^2}$

$$u_t - 2u_x = t,$$
 $u(x, 0) = \frac{1}{1 + x^2}$

Step 1: Set up the characteristic system of differential equations:

$$\frac{dx}{dt} = \frac{du}{dt} =$$

$$u_t - 2u_x = t,$$
 $u(x, 0) = \frac{1}{1 + x^2}$

Step 1: Set up the characteristic system of differential equations:

$$\frac{dx}{dt} = \frac{du}{dt} =$$

Step 2: Solve and express the constants of integration using the variables.

 $\implies x(t) = \qquad \qquad u(t) = \\ C = \qquad \qquad D =$

$$u_t - 2u_x = t,$$
 $u(x, 0) = \frac{1}{1 + x^2}$

Step 1: Set up the characteristic system of differential equations:

$$\frac{dx}{dt} = \frac{du}{dt} =$$

Step 2: Solve and express the constants of integration using the variables.

$$\implies x(t) = \qquad \qquad u(t) = \\ C = \qquad \qquad D =$$

Step 3: If possible, solve D = f(C) for u.

Step 4: Determine *f* and thus *u* using the initial values.

$$u_x + y^2 u_y = u^2,$$
 $u(x,1) = 1$ $x \in \mathbb{R}, y \in \mathbb{R}^+$

Auxiliary problem: U

$$u_x + y^2 u_y = u^2, \qquad u(x,1) = 1 \qquad x \in \mathbb{R}, \ y \in \mathbb{R}^+$$

Auxiliary problem: $U_x + y^2 U_y + u^2 \cdot U_u = 0$

$$u_x + y^2 u_y = u^2$$
, $u(x, 1) = 1$ $x \in \mathbb{R}, y \in \mathbb{R}^+$

Set up the characteristic system of differential equations:

$$\frac{dy}{dx} =$$
, $\frac{du}{dx} =$

Solve, express integration constants using the variables $\int \frac{dy}{y^2} = \int dx , \Longrightarrow -\frac{1}{y} = x - C_1 \Longrightarrow \boxed{C_1 = \frac{1}{y} + x}$ Completely analogously, we obtain $\boxed{C_2 = \frac{1}{u} + x}$

If possible, solve D = f(C) for u.

Solution satisfies:
$$\Phi(C_1, C_2) = \Phi(\frac{1}{y} + x, \frac{1}{u} + x) = 0$$

In the case of solvability, the following holds: $C_2 = f(C_1)$

 $\frac{1}{u} + x =$

If possible, solve D = f(C) for u.

Solution satisfies:
$$\Phi(C_1, C_2) = \Phi(\frac{1}{y} + x, \frac{1}{u} + x) = 0$$

In the case of solvability, the following holds: $C_2 = f(C_1)$

$$\frac{1}{u} + x = f(\frac{1}{y} + x) \Longrightarrow \frac{1}{u} = f(\frac{1}{y} + x) - x$$

Determine f and thus u using the initial values.

Initial condition $u(x,1) \stackrel{!}{=} 1$:

$$u(x,y) = \frac{1}{f\left(\frac{1}{y} + x\right) - x} \stackrel{y=1}{\Longrightarrow}$$

Example 3:

Determine the solution of the initial value problem.

$$u_t + 2xu_x = tu, \qquad x \in \mathbb{R}, \ t \in \mathbb{R}^+, u(x, 0) = \sin(x).$$

Solution: For the characteristics, the following holds

 $\begin{aligned} \frac{dx}{dt} &= 2x \implies \\ x(t) &= c_1 e^{2t}, \qquad c_1 = x e^{-2t}. \end{aligned}$

On the characteristic curves, the following holds

$$\frac{du}{dt} = tu \implies$$
$$u(x(t), t) = c_2 e^{\frac{t^2}{2}} \implies$$

So $c_2 = ue^{-\frac{t^2}{2}}$ With a suitable Φ then holds: $\Phi(c_1, c_2) = 0 \implies$

If solvable according to c_2 : $c_2 = f(c_1)$: $c_2 =$

On the other hand $u(x, 0) \stackrel{!}{=} \sin(x)$, therefore

u(x, 0) =

Thus, we obtain the solution.

$$u(x,t)=\sin(xe^{-2t})\cdot e^{\frac{t^2}{2}}.$$

Determine the solution u(x, t) of the following initial-boundary value problem.

$$u_t + 2u_x + u = 0, \quad x, t > 0$$

 $u(x, 0) = 0 \quad (x \ge 0)$
 $u(0, t) = t^2 \quad (t \ge 0)$

by means of the method of characteristics.

Idea: Determine the solution u(x, t) of the following initial-boundary value problem.

$$u_t + 2u_x + u = 0, \quad x, t > 0$$

 $u(x, 0) = 0 \quad (x \ge 0)$
 $u(0, t) = t^2 \quad (t \ge 0)$

by means of the method of characteristics.

Idea: Determine a solution for each initial condition u(x, 0) = 0, and one for the boundary condition $u(0, t) = t^2$ and smoothly assemble these solutions.

Using the method of characteristics, we obtain: $\frac{dx}{dt} = 2$, $\frac{du}{dt} = -u$

$$x(t) = 2t + C_1$$
 $u(x(t), t) = C_2 e^{-t}$

$$\implies C_1 = x - 2t, \ C_2 = ue^t$$

 $\Phi(x - 2t, ue^t) = 0$ yields a solution if solvable

$$u=e^{-t}f(x-2t).$$

From the boundary values $u(0, t) = t^2$ we obtain

$$t^2 = e^{-t}f(0-2t)$$

or with
$$\mu = -2t$$
 i.e. $t = -\frac{\mu}{2}$

$$f(\mu) = \left(-rac{\mu}{2}
ight)^2 e^{-rac{\mu}{2}}$$

and thus one obtains for the solution

$$u_R(x,t) = e^{-t}f(x-2t) = e^{-t}\left(-\frac{x-2t}{2}\right)^2 e^{\frac{-(x-2t)}{2}} = e^{-\frac{x}{2}}\left(\frac{2t-x}{2}\right)^2$$

On the other hand, if one substitutes into the general solution $u(x, t) = e^{-t}\tilde{f}(x - 2t)$ Substituting the initial data u(x, 0) = 0, we obtain:

The solutions can be smoothly combined along curves where $u = \tilde{u}$ holds. This is the case along the line x = 2t. Therefore, we obtain as the solution to the original problem:

$$u(x,t) = \begin{cases} e^{-\frac{x}{2}} \left(\frac{2t-x}{2}\right)^2 & x \le 2t \\ 0 & x > 2t \end{cases}$$

It would still need to be verified whether the solution along the line x = 2t is continuously (partially) differentiable.