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Advise on the exam for differential equations II for students of engineering

This file is intended to facilitate the preparation for the exam. We list important topics which does NOT mean that we exclude other subjects for the exam.

Publication of this file is [redacted] prohibited!

This version is based on the German original by Dr. Hanna Peywand Kiani

Absolutely necessary techniques

- Solve simple ordinary DEs, for example

  - separable (compare, e.g., method of characteristics)
  - linear with constant coefficients (compare, e.g., heat equation)

$$\dot{y}(t) + \alpha y(t) = h(t)$$

General solution of homogeneous DE: $y_h(t) = \gamma e^{-\alpha t}$, $\gamma \in \mathbb{R}$

Particular solution of inhomogeneous DE:
$y_p(t) := \gamma(t) e^{-\alpha t} \stackrel{\text{DE}}{\longrightarrow} \dot{\gamma}(t) \longrightarrow \gamma(t) \longrightarrow y_p(t)$

$y(t) = y_h(t) + y_p(t) = \gamma e^{-\alpha t} + y_p(t)$
Determine γ by initial values!

- Very simple integrals, integration by parts

  For the computation of, for example, d'Alembert, Fourier coefficients, characteristics

- Computation of Fourier coefficients (Sheets 6)

- polar $<--->$ Cartesian (Work sheet 4, Homework 4)

- Determinant/Eigenvalues of 2x2 matrices (Work sheet 4)

Sheet 1:

∅ not suitable for exams
xxx regularly in exams, many points
xx often in exams, intermediate number of points
x sometimes in exams as a short exercise

W1: Ansatz for the solution of the heat equation is given. Determine parameters by plugging it into the equation. ∅ Introduction

W2: Ansatz for the solution of the telegraph equation is given. Determine parameters by plugging it into the equation. ∅ Introduction

H1: Definitions, order, semi-, quasi-, linear etc. x

H2: Traffic model, continuity equation, transport equation $u_t - cu_x = 0$ ∅

Sheet 2: Method of characteristics

W1, W2, H1: Method of characteristics (standard), IVP:

$$u_t + a(x,t,u)u_x = b(x,t,u)$$ XXX

W3: First nonlinear DE, Questions about characteristics:
Shape of the characteristic?/ Is the space filled?

we know a more systematic way

H2: Questions about characteristics:
Shape/ solution constant along characteristics? XX

H3: Method of characteristics on quadrant ∅

Sheet 3:

W1: Burgers equation: two rarefaction waves, two shock waves, respectively xxx

W2: Continuity equation, $u_t + \left(\frac{(u-2)^4}{2}\right)_x = 0$, $u(x, 0)$ not continuous. xxx
jump condition, entropy solution.

W3: Nice-to-have. Irreversibility of non-smooth solutions of Burgers equation ∅

H1a) Short question about: Entropy solution, weak solution, rarefaction waves. x

H1b) Entropy solutions for continuity equations other than Burgers equation. xxx

H2) Burgers equation: rarefaction wave next to shock wave

- Required: solution up to the point where the waves meet xxx
- Nice-to-have: solution after the two waves met. ∅

H3) Traffic flow model, non convex flow function ϕ

Second order differential equations

Sheet 4:

P1) Transform to integrable form. Solve new DE

$(v(\alpha,\mu))_{\alpha,\mu}=0$

$\Rightarrow v(\alpha,\mu)=\Phi(\alpha)+\Psi(\mu)$

P2a) Harmonic functions.
P2b) Mean value property.

P3) Rotational symmetric solution of Laplace equation. Fundamental solutions

H1) Classify (parabolic, elliptic, hyperbolic)

H2) Mean value property/ maximum principle /determine values at given points by the uniqueness of the solution

Sheet 5:

P) IBVP: Inhomogeneous heat equation, inhomogeneous boundary values

- Transform to IBVP with homogeneous boundary values.

- Homogeneous DE, homogeneous boundary values: Product ansatz
  $\longrightarrow$ closed-form solution
  $\longrightarrow$ equating coefficients.

- Inhomogeneous DE, homogeneous boundary values: closed-form solution
  $\longrightarrow$ ordinary DE
  $\longrightarrow$ solve initial value problem for ordinary DE

- solution of original differential equation : superposition

H1) Laplace equation on discs: Product ansatz. $\emptyset$

$\longrightarrow$ closed-form solution

$\longrightarrow$ equating coefficients.

H2) Heat equation,

Derive solution approach for Neumann boundary conditions $\emptyset$

Solve example by equating coefficients.

Sheet 6:

W1) IVP wave equation, inhomogeneous,
Ansatz for the solution of the inhomogeneous DE is given
Homogeneous differential equation : d'Alembert
Combine (superposition).

xx

W2) IBVP wave equation, homogeneous DE, homogeneous boundary values
$\longrightarrow$ closed-form solution
$\longrightarrow$ equating coefficients ($A_K = 0$) und Fourier coefficients for B_k.

xxx

H1a) IVP wave equation, homogeneous DE: d'Alembert, standard xx

H1b) IVP, homogeneous DE, data not smooth: d'Alembert
Sketch solution for fixed values of t.

$\emptyset$

H2) IBVP wave equation, homogeneous DE, homogeneous boundary values
$\longrightarrow$ closed-form solution
$\longrightarrow$ equating coefficients ($B_K = 0$) and Fourier coefficients for A_k.

xxx

H3) IBVP: Inhomogeneous wave equation, inhomogeneous boundary values

don't need to solve but ↓

$\longrightarrow$ Reduce/transform to IBVP with homogeneous boundary values.

Summary of some
(not all)
closed-form formulas

No guarantee!

Please compare to the lecture notes/formulary before the exam!

Summary of some (not all) closed-form formulas for differential equations of first order:

Method of characteristics

$$u_t(x,t) + a(x,t,u)\, u_x(x,t) = b(x,t,u)\,.$$

Auxiliary problem : $U_t + a \cdot U_x + b \cdot U_u = 0$

$$\frac{dt}{ds} = 1 \qquad \frac{dx}{ds} = a(x,t,u), \qquad \frac{du}{ds} = b(x,t,u)$$

or (with t as parameter)

$$\frac{dx}{dt} = a(x,t,u)\,, \qquad \frac{du}{dt} = b(x,t,u). \tag{1}$$

Solving/integrating yields $C_1(x,t,u), C_2(x,t,u)$

Let $C_2 = f(C_1)$. Solve for u if possible

and determine f by the initial condition

Burgers and similar equations, rarefaction and shock waves

$$u_t + (f(u))_x = 0\,.$$

$$\frac{dx}{dt} = f'(u)\,, \qquad \frac{du}{dt} = 0. \tag{2}$$

the slope of the characteristics depends on u only

u is constant along characteristic

characteristics are straight lines

often sketches are useful

For (Riemann problem)

$$u_t + (f(u))_x = 0, (f \text{ strictly convex}), \qquad u(x,0) = \begin{cases} u_l & x \leq x_0 \\ u_r & x > x_0 \end{cases}$$

Entropy solution:

- If $u_l > u_r$: shock front (discontinuity curve) $s(t)$ with:

  Rankine- Hugoniot- jump condition:

$$\boxed{\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} =: \frac{[f]}{[u]}}$$

$$u(x,t) = \begin{cases} u_l & x \leq s(t), \\ u_r & x > s(t)\,. \end{cases}$$

- If $u_l < u_r$: rarefaction wave.
  Let $g = (f')^{-1}$ = inverse function of f'

$$u(x,t) = \begin{cases} u_l & x \leq x_0 + f'(u_l) \cdot t, \\ g\left(\dfrac{x - x_0}{t}\right) & x_0 + f'(u_l) \cdot t < x < x_0 + f'(u_r) \cdot t \\ u_r & x \geq x_0 + f'(u_r) \cdot t\,. \end{cases}$$

Summary of some (not all) closed-form formulas for **differential equations of second order**

For the exam:
Use the formulas that were derived in the lecture/auditorium exercise directly. Don't start from the product ansatz.

In the following we give solution formulas for:

Heat equation initial boundary value problem

Wave equation initial value problem

Wave equation initial boundary value problem

Laplace equation: rotational symmetric

Laplace equation: not rotational symmetric

I) Heat equation, initial boundary value problem (IBVP), homogeneous differential equation, homogeneous boundary values:

$$u_t - cu_{xx} = 0 \qquad c > 0,\ x \in (0, L),\ t > 0,$$

$$u(x, 0) = u_0(x) \qquad x \in [0, L],$$

$$u(0, t) = 0 \qquad t > 0,$$

$$u(L, t) = 0 \qquad t > 0,$$

$$u(x, t) = \sum_{k=1}^{\infty} a_k e^{-c\omega^2 k^2 t} \sin(k\omega x) \qquad \omega = \frac{\pi}{L}$$

$$u(x, 0) = \sum_{k=1}^{\infty} a_k \sin(k\omega x) \stackrel{!}{=} u_0(x)$$ equating coefficients may be possible

$$a_k = \frac{2}{L} \int_0^L u_0(x) \sin(k\omega x)\, dx$$ $\longleftarrow$ if equating coefficients not possible

II) Heat equation, IBVP, inhomogeneous differential equation, homogeneous boundary values:

$$u_t - c u_{xx} = h(x,t), \qquad x \in (0,L),\ t > 0, c > 0$$

$$u(x,0) = u_0(x), \qquad x \in (0,L)$$

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t > 0$$

$$u(x,t) = \sum_{k=1}^{\infty} a_k(t)\,\sin(k\omega x) \qquad \omega = \frac{\pi}{L}$$

Solve initial value problems

$$\dot{a}_k(t) + a_k(t)\,\frac{ck^2\pi^2}{L^2} = c_k(t), \quad a_k(0) = b_k$$

Where

$$\sum_{k=1}^{\infty} c_k(t) \sin(k\omega x) \stackrel{!}{=} h(x,t)$$ equating coefficients may be possible

otherwise:

$$c_k(t) = \frac{2}{L} \int_0^L h(x,t) \sin(k\omega x) dx$$

$$u(x,0) = \sum_{k=1}^{\infty} b_k \sin(k\omega x) \stackrel{!}{=} u_0(x)$$ equating coefficients may be possible

otherwise:

$$b_k = \frac{2}{L} \int_0^L u_0(x) \sin(k\omega x) dx$$

III) Heat equation, IBVP, inhomogeneous boundary values:

$$u_t - c\, u_{xx} = h(x,t), \qquad x \in (0,L),\ t > 0$$

$$u(x,0) = u_0(x), \qquad x \in (0,L)$$

$$u(0,t) = f(t) \qquad u(L,t) = g(t) \qquad t > 0$$

Reduce to solving a IBVP with homogeneous boundary values:

$$v(x,t) = u(x,t) - f(t) - \frac{x}{L}(g(t) - f(t))$$

yields a new problem for v with homogeneous boundary values.

In case the new DE is homogeneous: Case I).
In case the new DE is inhomogeneous: Case II).

Wave equation:

A) IVP, homogeneous:

$\tilde{u}_{tt} - c^2\tilde{u}_{xx} = 0$, $\tilde{u}(x,0) = g(x)$, $\tilde{u}_t(x,0) = h(x)$, $x \in \mathbb{R}$, $c > 0$

$$\tilde{u}(x,t) = \frac{1}{2}\,[g(x+ct) + g(x-ct)] + \frac{1}{2c}\int_{x-ct}^{x+ct} h(\alpha)\,d\alpha$$

B) IVP, inhomogeneous:

$u_{tt} - c^2 u_{xx} = f(x,t)$, $u(x,0) = g(x)$, $u_t(x,0) = h(x)$, $x \in \mathbb{R}$, $c > 0$

$$u(x,t) = \tilde{u} + \hat{u} \qquad (\tilde{u} \text{ as in A})$$

$$\hat{u}(x,t) = \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} f(\omega,\tau)\,d\omega\,d\tau$$

B) IBVP, homogeneous differential equation, homogeneous boundary values:

$$u_{tt} - c^2 u_{xx} = 0, \qquad x \in (0, L),\ t > 0,\ c > 0$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = w_0(x) \qquad x \in (0, L)$$

$$u(0,t) = 0 \qquad u(L,t) = 0 \qquad t > 0 \qquad \omega := \frac{\pi}{L}$$

$$u(x,t) \ = \ \sum_{k=1}^{\infty} \left[A_k \, \cos\left(ck\omega t\right) \ + \ B_k \, \sin\left(ck\omega t\right)\right] \, \sin\left(k\omega x\right)$$

Equating coefficients may be possible

$$u(x,0) \ = \ \sum_{k=1}^{\infty} A_k \, \sin\left(k\omega x\right) \ \stackrel{!}{=} u_0(x)$$

$$u_t(x,0) \ = \ \sum_{k=1}^{\infty} \ ck\omega \cdot B_k \, \sin\left(k\omega x\right) \stackrel{!}{=} w_0(x).$$

Otherwise:

$$A_k = \frac{2}{L}\int_0^L u_0(\alpha)\sin(k\omega\alpha)\,d\alpha, \quad B_k = \frac{2}{ck\pi}\int_0^L w_0(\alpha)\sin(k\omega\alpha)\,d\alpha$$

or $B_k = \dfrac{1}{ck\omega} b_k$ where $b_k = \dfrac{2}{L}\displaystyle\int_0^L w_0(\alpha)\sin(k\omega\alpha)\,d\alpha,$

C) **Inhomogeneous** differential equation, **homogeneous** boundary values

too complex/long for the exam

$$
\begin{aligned}
u_{tt} - c^2 u_{xx} &= h(x,t) \qquad && c > 0,\ x \in (0,L),\ t > 0 \\
u(x,0) &= u_0(x) && x \in (0,L), \\
u_t(x,0) &= v_0(x) && x \in (0,L), \\
u(0,t) &= 0 && t > 0, \\
u(L,t) &= 0 && t > 0,
\end{aligned}
$$

where $\omega = \frac{\pi}{L}$

$$
\boxed{u(x,t) = \sum_{k=1}^{\infty} q_k(t) \sin(k\omega x)}
$$

Solve $q_k''(t) + c^2k^2\omega^2 q_k(t) \;=\; c_k(t)\,, \qquad q_k(0) = a_k,\; q_k'(0) = b_k$

with: $a_k = \dfrac{2}{L}\displaystyle\int_0^L u_0(x)\sin(k\omega x)\,dx.$

$$b_k = \frac{2}{L}\int_0^L v_0(x)\sin(k\omega x)\,dx.$$

$$c_k(t) = \frac{2}{L}\int_0^L h(x,t)\sin(k\omega x)\,dx.$$

Fourier coefficients may be computed by equating coefficients!

D) IBVP, inhomogeneous boundary values:

$$u_{tt} - c^2 u_{xx} = h(x,t), \qquad x \in (0,L),\ t > 0$$

$$u(x,0) = u_0(x), \qquad u_t(x,0) = v_0(x) \qquad x \in (0,L)$$

$$u(0,t) = f(t) \qquad u(L,t) = g(t) \qquad t > 0$$

Reduce to the solution of IBVP with homogeneous boundary values:

$$w(x,t) = u(x,t) - f(t) - \frac{x}{L}(g(t) - f(t))$$

this yields a new problem for w with homogeneous boundary values.

In case that the new DE is a homogeneous wave equation: Case B)
In case that the new DE is an inhomogeneous wave equation: Case C)

Laplace equation

A) Rotational symmetric

Each rotational symmetric harmonic function on $\mathbb{R}^n \setminus \{0\}$ can be represented by the fundamental solution $\Phi(\boldsymbol{x})$ as

$$u(\boldsymbol{x}) = a\phi(\boldsymbol{x}) + c, \qquad a,\, c \in \mathbb{R}.$$

For $n = 2$: $\qquad \phi(x, y) = -\frac{1}{2\pi} \ln(\| \binom{x}{y} \|_2) = -\frac{1}{2\pi} \ln(\sqrt{x^2 + y^2})$

for $n = 3$: $\qquad \phi(x, y, z) = \frac{1}{4\pi} \|(x, y, z)\|_2^{-1} = \frac{1}{4\pi} \frac{1}{\sqrt{x^2+y^2+z^2}}.$

a and c can be determined by the boundary values.

B) Not rotational symmetric on rings, inside and outside of disks

Laplace operator in polar coordinates: $x = r\cos(\phi)$, $y = r\sin(\phi)$.

$$\Delta u = 0 \overset{r \neq 0}{\iff} u_{rr} + \tfrac{1}{r} u_r + \tfrac{1}{r^2} u_{\phi\phi} = 0 \,.$$

$$u(r,\phi) = c_0 + d_0 \ln(r) + \sum_{k=1}^{\infty} (c_k r^{-k} + d_k r^k)(a_k \cos(k\phi) + b_k \sin(k\phi))$$

Depending on the domain we need to exclude summand that are not bounded.

Procedure in the outer space:

$\Delta\, u = \; 0$ for $(x^2 + y^2 =)\, r^2 > R^2$ and $u(R, \phi) = u_0(\phi)$:

Since the solutions should stay bounded: $d_k = 0\,, \quad \forall k$.

We are left with: $u(r, \phi) = \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{\infty} r^{-k}\,(a_k \cos(k\phi)\; +\; b_k \sin(k\phi))$

Moreover, the boundary condition needs to be fulfilled $u(R, \phi) = u_0(\phi)$.

We obtain the solution

$$u(r, \phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{R}{r}\right)^k \,(A_k \cos(k\phi)\; +\; B_k \sin(k\phi))$$

where the Fourier coefficients of u_0 are

$$A_k = \frac{1}{\pi} \int_0^{2\pi} u_0(\phi) \cos(k\phi) \, d\phi$$

$$B_k = \frac{1}{\pi} \int_0^{2\pi} u_0(\phi) \sin(k\phi) \, d\phi$$

Procedure in the inner space:

$\Delta\, u = \ 0$ für $(x^2 + y^2 =)\, r^2 < R^2$ and $u(R, \phi) = u_0(\phi)$:

Since the solutions should stay bounded : $d_0 = 0, c_k = 0\,, \quad \forall k$.

We are left with: $u(r, \phi) = \dfrac{a_0}{2} + \displaystyle\sum_{k=1}^{\infty} r^k\, (a_k \cos(k\phi) \ + \ b_k \sin(k\phi))$

Moreover, the boundary condition needs to be fulfilled $u(R, \phi) = u_0(\phi)$.

We obtain the solution

$$u(r, \phi) = \frac{A_0}{2} + \sum_{k=1}^{\infty} \left(\frac{r}{R}\right)^k \ (A_k \cos(k\phi) \ + \ B_k \sin(k\phi))$$

where the Fourier coefficients of u_0 are

$$A_k = \frac{1}{\pi} \int_0^{2\pi} u_0(\phi) \cos(k\phi) \, d\phi$$

$$B_k = \frac{1}{\pi} \int_0^{2\pi} u_0(\phi) \sin(k\phi) \, d\phi$$