Mathematics Department
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SuSe 2025

Exam Differential Equations II

27. August 2025

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Task	Points	Evaluator
1		
2		
3		
4		

$\sum =$

Problem 1: [7 Points]

Compute the solution of the following initial value problem for $u(x,t)$:

$$u_t + (2t+1)\,u_x = -ut\,, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+,$$
$$u(x,0) = \sin(2x) \qquad x \in \mathbb{R}\,.$$

Solution:

With the characteristics method one computes:

$\frac{dx}{dt} = 2t+1 \Longrightarrow dx = (2t+1)dt \Longrightarrow x = t^2 + t + C$ **[1 point]**

$$\frac{du}{dt} = -ut \Longrightarrow \frac{du}{u} = -tdt \Longrightarrow \ln(|u|) = -\frac{t^2}{2} + \tilde{D}$$
$$\Longrightarrow |u| = \pm e^{-\frac{t^2}{2}+\tilde{D}} \Longrightarrow u = De^{-\frac{t^2}{2}}$$ **[1 point]**

With $C = x - t^2 - t$ and $D = ue^{\frac{t^2}{2}}$ using the

ansatz $D = f(C)$

we obtain

$ue^{\frac{t^2}{2}} = f(x - t^2 - t)$ **[2 points]**

and the general solution:: $u(x,t) = e^{-\frac{t^2}{2}} f(x - t^2 - t)$. **[1 point]**

The initial condition requires:

$u(x,0) = e^{-\frac{0^2}{2}} f(x - 0^2 - 0) = f(x) \stackrel{!}{=} \sin(2x)\,.$ **[1 point]**

Hence the solution of the IVP is

$u(x,t) = e^{-\frac{t^2}{2}} \sin(2x - 2t^2 - 2t).$ **[1 point]**

Problem 2: [4 Points]

Determine the entropy solution of the differential equation

$$u_t + (f(u))_x = 0$$

with the flux function

$$f(u) = \Big(\frac{u-1}{3}\Big)^2$$

and the initial condition

$$u(x,0) = \begin{cases} 2 & x \le 0, \\ -1 & 0 < x. \end{cases}$$

Note: Only the solution for the given initial values is required. You don't need to give solutions for general initial values!

Solution:

An ambiguity of the solution obtained using the methods of characteristics arises immediately (i.e. already at $t = 0$). A shock front $s(t)$ must be introduced with $u_l = 2$ and $u_r = 1$ **[1 point]**

With $f(u_l) = \left(\frac{2-1}{3}\right)^2 = \frac{1}{9}$, $\quad f(u_r) = \left(\frac{-1-1}{3}\right)^2 = \frac{4}{9}$ **[1 point]**

we obtain

$$\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{1}{9} - \frac{4}{9}}{2-(-1)} = -\frac{1}{9}$$ **[1 point]**

and thus

$$u(x,t) = \begin{cases} u_l = 2 & x \le s(t) = -\frac{t}{9} \\ u_r = -1 & -\frac{t}{9} < x. \end{cases}$$ **[1 point]**

Problem 3: [2 points]

Let $u(x, y)$ be the solution to the following boundary value problem

$$\begin{aligned} \Delta u &= u_{xx} + u_{yy} = 0, \quad \text{in } \Omega := \left\{ \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2,\, 0 < x < 20,\, 0 < y < 25 \right\}, \\ u(x, y) &= 10, \quad \text{on } \partial\Omega\,. \end{aligned}$$

Determine the solution u without calculation. Justify your answer.

Solution:

$u(x, y)$ is constant on the boundary of Ω. Since the maximum and minimum values of u are attained on the boundary u is constant on the entire rectangel Ω and the solution is

$$u(x, y) = 10 \qquad \forall (x, y) \in \Omega.$$

Alternative solution: The constant function $u(x, y) = 10$ solves the Laplace equation on the whole rectangle Ω and fulfills the boundary condition. Since the solution is unique, we obtain

$$u(x, y) = 10 \qquad \forall (x, y) \in \Omega.$$

Problem 4: [7 points]

a) Consider the initial boundary value problem

$$\begin{aligned} u_t - u_{xx} &= \frac{x}{\pi}\cos(t) && \text{for } x \in (0,\pi), t > 0, \\ u(x,0) &= 1 - \frac{x}{\pi} + 2\Big(\sin(x) - \sin(3x)\Big) && \text{for } x \in [0,\pi], \\ u(0,t) &= 1, && \text{for } t > 0, \\ u(\pi,t) &= \sin(t) && \text{for } t > 0. \end{aligned}$$

Introduce a suitable function v in order to convert the problem into an initial boundary value problem with homogeneous boundary conditions for v.

Determine the differential equation and the initial conditions for v.

b) Solve the following initial boundary value problem:

$$\begin{aligned} v_t - v_{xx} &= 0 && \text{for } x \in (0,\pi), t > 0, \\ v(x,0) &= 2\sin(x) - 2\sin(3x) && \text{for } x \in [0,\pi], \\ v(0,t) = 0, &\quad v(\pi,t) = 0 && \text{for } t > 0. \end{aligned}$$

c) Give the solution to the initial boundary value problem from part a).

Solution:

a) Convert the problem into an initial boundary value problem with homogeneous boundary conditions:

With

$v(x,t) = u(x,t) - 1 - \frac{x}{\pi}(\sin(t) - 1) =$

we get

$u(x,t) = v(x,t) + 1 + \frac{x}{\pi}(\sin(t) - 1)\,.$ **[1 point]**

and obtain

$u_t = v_t + \frac{x}{\pi}\cos(t), \qquad v_{xx} = u_{xx}$

New PDE: $v_t + \frac{x}{\pi}\cos(t) - v_{xx} = \frac{x}{\pi}\cos(t) \iff$

$\boxed{v_t - v_{xx} = 0}$ **[1 point]**

Initial values:

$$\begin{aligned} v(x,0) &= u(x,0) - 1 - \frac{x}{\pi}(\sin(t) - 1) \\ &= 1 - \frac{x}{\pi} + 2\Big(\sin(x) - \sin(3x)\Big) - 1 + \frac{x}{\pi} \iff \end{aligned}$$

$\boxed{v(x,0) = 2\sin(x) - 2\sin(3x)}$ **[1 point]**

Boundary values: $\boxed{v(0,t) = v(\pi,t) = 0}$

b) Using the ansatz

$$v(x,t) \;=\; \sum_{k=1}^{\infty} a_k e^{-c\omega^2 k^2 t} \sin(k\omega x) \;=\; \sum_{k=1}^{\infty} a_k e^{-k^2 t} \sin(kx) \qquad \textbf{(1 point)}$$

with $\omega = \frac{\pi}{\pi} = 1$ and $c = 1$, the initial condition reads

$v(x,0) \;=\; \sum_{k=1}^{\infty} a_k \sin(kx) \;\stackrel{!}{=}\; 2\sin(x) \;-\; 2\sin(3x)\,.$

Comparison of the coefficients gives

$\Longrightarrow\; a_1 = 2,\; a_3 = -2,\; a_k = 0 \qquad \forall k \notin \{1,3\}.$

Hence we obtain

$$v(x,t) \;=\; 2\,e^{-t}\,\sin(x) \;-\; 2\,e^{-9t}\,\sin(3x) \qquad [\textbf{2 points}]$$

c) For the solution of a) we thus get

$$\begin{aligned} u(x,t) &=\; v(x,t) \,+\, 1 + \frac{x}{\pi}(\sin(t) - 1) \\ &=\; 2\,e^{-t}\,\sin(x) \;-\; 2\,e^{-9t}\,\sin(3x) \;+\; 1 + \frac{x}{\pi}(\sin(t) - 1)\,. \qquad [\textbf{1point}] \end{aligned}$$