Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
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Differential Equations II for Engineering Students

Work sheet 1

Exercise 1: (Repetition of DGL I)

a) Let λ be any fixed real number. Determine a real representation of the general solution to the differential equation
$$y''(t) - \lambda\, y(t) = 0\,.$$

b) Let L be another fixed positive real number. Determine all solutions to the boundary value problem
$$y''(t) - \lambda\, y(t) = 0 \quad y(0) = y(L) = 0\,.$$

For which $\lambda \in \mathbb{R}$ does the boundary value problem have nontrivial solutions?

The λ-values for which there exist non-trivial solutions (i.e. solutions that are not constantly equal to zero) are called eigenvalues of the problem. The corresponding solutions are called eigenfunctions.

Remark: *The solutions to this eigenvalue problem will be needed again and again during the semester!*

Solution hints for the exercise 1:

a) Following DGL I, we calculate the characteristic polynomial : $\mu^2 - \lambda = 0$ with the zeros
$$\mu_{1,2} = \pm\sqrt{\lambda} \quad \Longrightarrow y(t) = \begin{cases} c_1\, e^{\sqrt{\lambda}t} + c_2\, e^{-\sqrt{\lambda}t} & \lambda > 0\,, \\ c_1 + c_2 t & \lambda = 0\,, \\ c_1 \cos(\sqrt{-\lambda}t) + c_2 \sin(\sqrt{-\lambda}t) & \lambda < 0\,. \end{cases}$$

b) For case $\lambda > 0$, from the boundary value for $t = 0$ it follows immediately that $c_2 = -c_1$. The boundary value at L yields:
$$c_1 \left(e^{\sqrt{\lambda}L} - e^{-\sqrt{\lambda}L}\right) = 0 \Longrightarrow c_1 \left(e^{2\sqrt{\lambda}L} - 1\right) = 0 \Longrightarrow c_1 = 0$$
In this case there exist only the trivial solution $y(t) = 0$

For case $\lambda = 0$, the solution is a linear function. The only linear function that exists in $t = 0$ and also disappears in $t = L > 0$ is again the trivial solution.

For case $\lambda < 0$, from the boundary value for $t = 0$ it follows immediately that $c_1 = 0$. The boundary value in L yields:
$$c_2 \sin(\sqrt{-\lambda}L) = 0 \Longrightarrow c_2 = 0 \vee \sqrt{-\lambda}L = k\pi$$
Hence, only for $\lambda = -\left(\frac{k\pi}{L}\right)^2, k \in \mathbb{N}$, we obtain the non-trivial solutions $y_k(t) = \sin\left(\frac{k\pi}{L}t\right)$.

Exercise 2:

Determine the appropriate real Fourier series for the following functions:

a) Odd $2L-$ periodic continuation of
$f : [0,1[\to \mathbb{R}, \quad f(t) = \sin(4\pi x) + 2\sin(6\pi x) \quad L = 1\,.$

b) Even $2L-$ periodic continuation of
$f : [0, \frac{\pi}{2}[\to \mathbb{R}, \quad L = \frac{\pi}{2}$ und

$$f(t) \;=\; \begin{cases} 2, & 0 \le t < \frac{\pi}{4}, \\ 0, & \frac{\pi}{4} \le t < \frac{\pi}{2}. \end{cases}$$

Determine the first four non-vanishing summands of the Fourier series.

Solution hint to Exercise 2:

a) Since the function $f(x)$ is continued oddly, a Fourier sine series is used. Since $2L$ is a period of the function, one chooses $2L-$ periodic sine functions. So we define a series in the form

$$F(x) \;=\; \sum_{k=1}^{\infty} b_k \,\sin\left(k\frac{2\pi}{2L}\,x\right)$$

$$L = 1 \implies F(x) \;=\; \sum_{k=1}^{\infty} b_k \,\sin\left(k\pi x\right)$$

Due to orthogonality relations between the $\sin(k\pi x)$ and $\sin(l\pi x)$ (see Mathe II) and by assuming that the Fourier series is as good as possible approximation of f , we have

$$b_4 = 1\,, \quad b_6 = 2\,, \quad b_k = 0 \quad \text{otherwise.}$$

b) Since the function $f(t)$ is continued evenly, a Fourier cosine series is used. Since $2L$ is a period of the function, one chooses $2L-$ periodic cosine functions.

$$F(t) \;=\; \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \,\cos\left(k\frac{2\pi}{\pi}\,t\right).$$

Since $T = \pi,$ we have for the coefficients

$$\begin{aligned} a_k &= \frac{4}{\pi}\int_0^{\frac{T}{2}} f(t)\cos\left(k\omega t\right)dt \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} 2\cdot\cos\left(k\frac{2\pi}{\pi}t\right)dt + \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 0\cdot\cos\left(k\frac{2\pi}{\pi}t\right)dt \\ &= \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \cos(2kt)dt \end{aligned}$$

For $k = 0$ it holds

$$a_0 = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} 1dt = \frac{8}{\pi}[t]_0^{\frac{\pi}{4}} = 2$$

and for $k > 0$ we obtain

$$a_k = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} \cos(2kt)dt = \frac{4}{\pi}\left[\frac{1}{k}\sin(2kt)\right]_0^{\frac{\pi}{4}} = \frac{4}{\pi k}\sin\left(\frac{k\pi}{2}\right).$$

Hence

$$a_k = \begin{cases} 2 & k = 0 \\ 0 & k = 2m,\ m \in \mathbb{N} \\ \frac{4(-1)^m}{\pi(2m+1)} & k = 2m+1,\ m \in \mathbb{N}_0 \end{cases}$$

so

$$a_0 = 2 \qquad a_1 = \frac{4}{\pi} \qquad a_3 = -\frac{4}{3\pi} \qquad a_5 = \frac{4}{5\pi} \cdots$$

The first four non-vanishing summands of the Fourier series are e.g.

$$1 + \frac{4}{\pi}\cos(2t) - \frac{4}{3\pi}\cos(6t) + \frac{4}{5\pi}\cos(10t).$$

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