Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
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Differential Equations II for Engineering Students

Homework sheet 1

Exercise 1: (Repetition Analysis II)

For the derivation of parameter-dependent integrals the **Leibniz rule**

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))$$

applies if f is sufficiently smooth.

a) Find the derivative of the function $F(x)$

$$F(x) := \int_{-x}^{x^2} e^{xt}\,dt$$

(i) by first integrating with respect to t and then deriving with respect to x,

(ii) by first deriving with respect to x and then integrating with respect to t.

b) Compute $\lim\limits_{x\to 0} F'(x)$.

Solution to Exercise 1:

a) For $x \neq 0$

(i)

$$\begin{aligned} F'(x) &= \frac{d}{dx}\left(\frac{e^{xt}}{x}\Bigg|_{-x}^{x^2} \right) = \frac{d}{dx}\left(\frac{e^{x^3} - e^{-x^2}}{x} \right) \\ &= \frac{(3x^2 e^{x^3} + 2xe^{-x^2})x - (e^{x^3} - e^{-x^2})}{x^2} = 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left(e^{x^3} - e^{-x^2}\right). \end{aligned}$$

(ii) $F(x) = \int\limits_{-x}^{x^2} e^{xt}\,dt\,, \qquad b(x) := x^2,\ a(x) := -x,\ f(t,x) := e^{xt}$

$$b'(x) = 2x \qquad a'(x) = -1$$
$$f(b(x),x) = e^{x^3} \qquad f(a(x),x) = e^{-x^2}$$

$$
\begin{aligned}
F'(x) &= \int\limits_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t,x)\,dt + b'(x) f(b(x),x) - a'(x) f(a(x),x) \\
&= \int\limits_{-x}^{x^2} t e^{xt}\,dt + 2x\,e^{x^3} + e^{-x^2} \\
&= \left[\frac{t}{x} e^{tx}\right]_{-x}^{x^2} - \int\limits_{-x}^{x^2} \frac{1}{x} e^{xt}\,dt + 2x\,e^{x^3} + e^{-x^2} \\
&= 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left[e^{tx}\right]_{-x}^{x^2} = 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left(e^{x^3} - e^{-x^2}\right)
\end{aligned}
$$

b) Using the rule of l'Hospital we get

$$
\begin{aligned}
F'(0) &= 3x\,e^{x^3} + 2e^{-x^2}\Big|_{x=0} - \lim_{x\to 0} \frac{1}{x^2}(e^{x^3} - e^{-x^2}) \\
&= 2 - \lim_{x\to 0} \frac{3x^2 e^{x^3} + 2xe^{-x^2}}{2x} = 2 - \lim_{x\to 0} \frac{3xe^{x^3} + 2e^{-x^2}}{2} \\
&= 2 - 1 = 1\,.
\end{aligned}
$$

Exercise 2:

The purpose of this exercise is to repeat the *differential operators*

$$\operatorname{div}, \quad \mathbf{grad}, \quad \mathbf{rot}, \quad \Delta, \quad \nabla$$

which are known from Analysis III.

Let $D \subset \mathbb{R}^3$ be an open set and $\boldsymbol{x} = (x_1, x_2, x_3)^\top \in D$. We consider the functions

- $\boldsymbol{f} : D \to \mathbb{R}^3$ mit $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x}))^\top$,
- $g : D \to \mathbb{R}$,

where both $\boldsymbol{f}$ and g are $\mathcal{C}^3$-functions.

(a) Indicate which of the following expressions is defined. If it is defined, identify whether the corresponding expression is a vector in $\mathbb{R}^3$ or a number in $\mathbb{R}$.

(i) $\operatorname{div}(\mathbf{grad}\, f)(\boldsymbol{x})$,

(ii) $\mathbf{grad}\,(\Delta g)(\boldsymbol{x})$,

(iii) $\mathbf{rot}\,(\operatorname{div} \boldsymbol{f})(\boldsymbol{x})$,

(iv) $\Delta\,(\operatorname{div} \boldsymbol{f})(\boldsymbol{x})$.

(b) Show the two equalities

$$\operatorname{div}(\mathbf{rot}\, f)(\boldsymbol{x}) = 0 \qquad \text{and} \qquad \mathbf{rot}\,(\nabla g)(\boldsymbol{x}) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) Show the following equality

$$\nabla(\operatorname{div} \boldsymbol{f})(\boldsymbol{x}) - \mathbf{rot}(\mathbf{rot}\, \boldsymbol{f})(\boldsymbol{x}) = \begin{pmatrix} \Delta f_1(\boldsymbol{x}) \\ \Delta f_2(\boldsymbol{x}) \\ \Delta f_3(\boldsymbol{x}) \end{pmatrix}.$$

Solution sketch of exercise 2:

(a) The first thing to notice is that the functions are sufficiently regular such that every expression has the required differentiability to be defined. Therefore, it is left to show if the differential operators can actually operate on the corresponding function with respect to the dimensions.

(i) The gradient $\mathbf{grad}$ is only defined for scalar functions mapping into $\mathbb{R}$. Thus, $\mathbf{grad}\,\boldsymbol{f}$ is not defined.

(ii) The Laplace operator Δ is also defined for scalar functions like $g : D \to \mathbb{R}$ and gives again a scalar function $\Delta g : D \to \mathbb{R}$. Hence, we can apply the gradient operator to this function which gives a vector valued function. In particular, for every $\boldsymbol{x} \in D$ the expression $\mathbf{grad}\,(\Delta g)(\boldsymbol{x})$ is a vector in $\mathbb{R}^3$.

(iii) The divergence div is defined for vector fields such that $\operatorname{div} \boldsymbol{f}$ makes sense as a function from D to $\mathbb{R}$. However, the $\mathbf{rot}$ operator is defined for functions mapping into $\mathbb{R}^3$ and therefore, $\mathbf{rot}\,(\operatorname{div} \boldsymbol{f})$ is *not* defined.

(iv) As argued in (iii), $\operatorname{div} \boldsymbol{f} : D \to \mathbb{R}$ is a scalar function such that we can apply the Laplacian. Moreover, for each $\boldsymbol{x} \in D$, the expression $\Delta\,(\operatorname{div} \boldsymbol{f})(\boldsymbol{x})$ is a number in $\mathbb{R}$.

(b) First, we remind that for functions $\varphi : D \to \mathbb{R}$ which are two times continuously differentiable, the *theorem of Schwarz* gives

$$\frac{\partial^2 \varphi}{\partial x_i \partial x_j} = \frac{\partial^2 \varphi}{\partial x_j \partial x_i}, \quad i, j = 1, 2, 3.$$

We will use this in the following for every component of the corresponding function.

We start with computing

$$
\begin{aligned}
\operatorname{div}(\,\mathbf{rot}\,\boldsymbol{f})(\boldsymbol{x}) &= \operatorname{div}\begin{pmatrix} \frac{\partial f_3}{\partial x_2}(\boldsymbol{x}) - \frac{\partial f_2}{\partial x_3}(\boldsymbol{x}) \\ \frac{\partial f_1}{\partial x_3}(\boldsymbol{x}) - \frac{\partial f_3}{\partial x_1}(\boldsymbol{x}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) - \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}) \end{pmatrix} \\
&= \frac{\partial}{\partial x_1}\left(\frac{\partial f_3}{\partial x_2}(\boldsymbol{x}) - \frac{\partial f_2}{\partial x_3}(\boldsymbol{x})\right) + \frac{\partial}{\partial x_2}\left(\frac{\partial f_1}{\partial x_3}(\boldsymbol{x}) - \frac{\partial f_3}{\partial x_1}(\boldsymbol{x})\right) + \frac{\partial}{\partial x_3}\left(\frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) - \frac{\partial f_1}{\partial x_2}(\boldsymbol{x})\right) \\
&= \frac{\partial^2 f_3}{\partial x_1 \partial x_3}(\boldsymbol{x}) - \frac{\partial^2 f_2}{\partial x_1 \partial x_3}(\boldsymbol{x}) + \frac{\partial^2 f_1}{\partial x_2 \partial x_3}(\boldsymbol{x}) - \frac{\partial^2 f_3}{\partial x_1 \partial x_2}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_1 \partial x_3}(\boldsymbol{x}) - \frac{\partial^2 f_1}{\partial x_2 \partial x_3}(\boldsymbol{x}) \\
&= 0.
\end{aligned}
$$

Moreover,

$$
\mathbf{rot}\,(\nabla g)\,(\,\boldsymbol{x}\,) = \mathbf{rot}\begin{pmatrix} \frac{\partial g}{\partial x_1} \\ \frac{\partial g}{\partial x_2} \\ \frac{\partial g}{\partial x_3} \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_2}\frac{\partial g}{\partial x_3} - \frac{\partial}{\partial x_3}\frac{\partial g}{\partial x_2} \\ \frac{\partial}{\partial x_3}\frac{\partial g}{\partial x_1} - \frac{\partial}{\partial x_1}\frac{\partial g}{\partial x_3} \\ \frac{\partial}{\partial x_1}\frac{\partial g}{\partial x_2} - \frac{\partial}{\partial x_2}\frac{\partial g}{\partial x_1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.
$$

(c) As before, we use Schwarz's theorem to compute:

$$
\begin{aligned}
\nabla\,(\operatorname{div}\boldsymbol{f})\,(\boldsymbol{x}) &= \nabla\left(\frac{\partial f_1}{\partial x_1}(\boldsymbol{x}) + \frac{\partial f_2}{\partial x_2}(\boldsymbol{x}) + \frac{\partial f_3}{\partial x_3}(\boldsymbol{x})\right) \\
&= \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1^2}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_1 \partial x_2}(\boldsymbol{x}) + \frac{\partial^2 f_3}{\partial x_1 \partial x_3}(\boldsymbol{x}) \\ \frac{\partial^2 f_1}{\partial x_1 \partial x_2}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_2^2}(\boldsymbol{x}) + \frac{\partial^2 f_3}{\partial x_2 \partial x_3}(\boldsymbol{x}) \\ \frac{\partial^2 f_1}{\partial x_1 \partial x_3}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_2 \partial x_3}(\boldsymbol{x}) + \frac{\partial^2 f_3}{\partial x_3^2}(\boldsymbol{x}) \end{pmatrix}. \qquad (1)
\end{aligned}
$$

If we denote

$$
\boldsymbol{r}\,(\,\boldsymbol{x}\,) = \mathbf{rot}\;\boldsymbol{f}\,(\,\boldsymbol{x}\,) = \begin{pmatrix} \frac{\partial f_3}{\partial x_2}(\boldsymbol{x}) - \frac{\partial f_2}{\partial x_3}(\boldsymbol{x}) \\ \frac{\partial f_1}{\partial x_3}(\boldsymbol{x}) - \frac{\partial f_3}{\partial x_1}(\boldsymbol{x}) \\ \frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) - \frac{\partial f_1}{\partial x_2}(\boldsymbol{x}) \end{pmatrix},
$$

we conclude

$$
\begin{aligned}
&\mathbf{rot}\,(\,\mathbf{rot}\,\boldsymbol{f})\,(\boldsymbol{x}) = \mathbf{rot}\,\boldsymbol{r}(\boldsymbol{x}) \\
&= \begin{pmatrix} \frac{\partial r_3}{\partial x_2}(\boldsymbol{x}) - \frac{\partial r_2}{\partial x_3}(\boldsymbol{x}) \\ \frac{\partial r_1}{\partial x_3}(\boldsymbol{x}) - \frac{\partial r_3}{\partial x_1}(\boldsymbol{x}) \\ \frac{\partial r_2}{\partial x_1}(\boldsymbol{x}) - \frac{\partial r_1}{\partial x_2}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \frac{\partial}{\partial x_2}\left(\frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) - \frac{\partial f_1}{\partial x_2}(\boldsymbol{x})\right) - \frac{\partial}{\partial x_3}\left(\frac{\partial f_1}{\partial x_3}(\boldsymbol{x}) - \frac{\partial f_3}{\partial x_1}(\boldsymbol{x})\right) \\ \frac{\partial}{\partial x_3}\left(\frac{\partial f_3}{\partial x_2}(\boldsymbol{x}) - \frac{\partial f_2}{\partial x_3}(\boldsymbol{x})\right) - \frac{\partial}{\partial x_1}\left(\frac{\partial f_2}{\partial x_1}(\boldsymbol{x}) - \frac{\partial f_1}{\partial x_2}(\boldsymbol{x})\right) \\ \frac{\partial}{\partial x_1}\left(\frac{\partial f_1}{\partial x_3}(\boldsymbol{x}) - \frac{\partial f_3}{\partial x_1}(\boldsymbol{x})\right) - \frac{\partial}{\partial x_2}\left(\frac{\partial f_3}{\partial x_2}(\boldsymbol{x}) - \frac{\partial f_2}{\partial x_3}(\boldsymbol{x})\right) \end{pmatrix} \\
&= \begin{pmatrix} \frac{\partial^2 f_2}{\partial x_1 \partial x_2}(\boldsymbol{x}) - \frac{\partial^2 f_1}{\partial x_2^2}(\boldsymbol{x}) - \frac{\partial^2 f_1}{\partial x_3^2}(\boldsymbol{x}) + \frac{\partial^2 f_3}{\partial x_1 \partial x_3}(\boldsymbol{x}) \\ \frac{\partial^2 f_3}{\partial x_2 \partial x_3}(\boldsymbol{x}) - \frac{\partial^2 f_2}{\partial x_3^2}(\boldsymbol{x}) - \frac{\partial^2 f_2}{\partial x_1^2}(\boldsymbol{x}) + \frac{\partial^2 f_1}{\partial x_1 \partial x_2}(\boldsymbol{x}) \\ \frac{\partial^2 f_1}{\partial x_1 \partial x_3}(\boldsymbol{x}) - \frac{\partial^2 f_3}{\partial x_1^2}(\boldsymbol{x}) - \frac{\partial^2 f_3}{\partial x_2^2}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_2 \partial x_3}(\boldsymbol{x}) \end{pmatrix}. \qquad (2)
\end{aligned}
$$

Combining equations (1) and (2), we obtain

$$
\nabla(\operatorname{div}\,\boldsymbol{f}\,)(\,\boldsymbol{x}\,) - \mathbf{rot}(\,\mathbf{rot}\;\boldsymbol{f}\,)(\,\boldsymbol{x}\,) = \begin{pmatrix} \frac{\partial^2 f_1}{\partial x_1^2}(\boldsymbol{x}) + \frac{\partial^2 f_1}{\partial x_2^2}(\boldsymbol{x}) + \frac{\partial^2 f_1}{\partial x_3^2}(\boldsymbol{x}) \\ \frac{\partial^2 f_2}{\partial x_1^2}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_2^2}(\boldsymbol{x}) + \frac{\partial^2 f_2}{\partial x_3^2}(\boldsymbol{x}) \\ \frac{\partial^2 f_3}{\partial x_1^2}(\boldsymbol{x}) + \frac{\partial^2 f_3}{\partial x_2^2}(\boldsymbol{x}) + \frac{\partial^2 f_3}{\partial x_3^2}(\boldsymbol{x}) \end{pmatrix} = \begin{pmatrix} \Delta f_1(\boldsymbol{x}) \\ \Delta f_2(\boldsymbol{x}) \\ \Delta f_3(\boldsymbol{x}) \end{pmatrix}.
$$

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