Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
Prof. Dr. T. Schmidt,
Dr. H. P. Kiani, J. Schütt

Differential Equations II for Engineering Students

Work sheet 5

Exercise 1:

Using a suitable product ansatz, solve the following Dirichlet boundary value problem for the Laplace equation in polar coordinates on the disk $r^2 = x^2 + y^2 \leq 9$.

$$\begin{aligned} r^2 u_{rr} + r u_r + u_{\varphi\varphi} &= 0 \qquad && 0 \leq r < 3 \\ u(3,\varphi) &= \cos^2(\varphi) && \varphi \in \mathbb{R}. \end{aligned}$$

Hints:

To solve Euler's equation
$r^2 \cdot g''(r) + ar \cdot g'(r) + b \cdot g(r) = 0$ use the ansatz $g(r) = r^k$.
It holds: $\cos^2(\varphi) = \dfrac{1}{2}\,(1 + \cos(2\varphi))$.

Exercise 2:

Let $P : \mathbb{R}^2 \to \mathbb{R}$ be a cubic polynomial of the form

$$\begin{aligned} P(x,y) \; &= a_1 x^3 + a_2 x^2 y + a_3 x y^2 + a_4 y^3 \\ &\quad + b_1 x^2 + b_2 xy + b_3 y^2 \\ &\quad + c_1 x + c_2 y + d, \end{aligned}$$

with $a_i, b_j, c_k, d \in \mathbb{R}$. Determine all coefficients for which P is harmonic on $\mathbb{R}^2$.

Exercise 3:

(a) Let $U \subset \mathbb{R}^n$ be a bounded open set with connected boundary ∂U, and $g : \partial U \to \mathbb{R}$ a continuous function. Let u be a continuous solution on the closure $\overline{U}$ of

$$\begin{cases} \Delta u = 0 & \text{on} \quad U, \\ u = g & \text{on} \quad \partial U. \end{cases}$$

Show: If g has no zeros, then u also has no zeros.

(b) Let

$$V = \{(x,y)^\top \in \mathbb{R}^2 \ : \ 0 < x^2 + y^2 < 1\}, \qquad \Gamma = \{(x,y)^\top \in \mathbb{R}^2 \ : \ x^2 + y^2 = 1\}.$$

Then the problem

$$\begin{cases} \Delta u = 0 & \text{on} \quad V, \\ u = 0 & \text{on} \quad \Gamma, \end{cases}$$

has the two solutions

$$u_1 = 0 \qquad \text{and} \qquad u_2(x,y) = \Phi(|(x,y)|),$$

where Φ denotes the fundamental solution of Laplace's equation.

Why does this *not* contradict the uniqueness theorem of the solution (Corollary 2 on page 70 of the lecture)?

(c) Let

$$W_1 = \{(x,y)^\top \in \mathbb{R}^2 \ : \ (x+2)^2 + y^2 < 1\}, \qquad W_2 = \{(x,y)^\top \in \mathbb{R}^2 \ : \ (x-2)^2 + y^2 < 1\},$$

and $W = W_1 \cup W_2$. Then the function

$$u(x,y) = \begin{cases} 1 & \text{fi¿½r } (x,y)^\top \in \overline{W_1}, \\ 2 & \text{fi¿½r } (x,y)^\top \in \overline{W_2}. \end{cases}$$

is harmonic on W, not constant, and assumes both its maximum and minimum also in the interior of W. Why does this *not* contradict the strong maximum principle?

Discussion: 17.06.-21.06.2024