Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
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Differential Equations II for Engineering Students

Work sheet 4

Exercise 1:

We are looking for a solution to the initial value problem

$$u_t + (u+2)u_x = 0 \qquad x \in \mathbb{R},$$
$$u(x,0) = \frac{1-x}{2} \qquad x \in \mathbb{R}$$

for $0 < t < t^*$ with a sufficiently small t^*.

a) Determine the solution to the initial value problem using the characteristic method.

b) In which intervals $]0, t^*[$ is the solution from a) defined?

c) Are the characteristic curves straight lines? Explain your answer.

d) Sketch the characteristics through the points $(x_0, 0)$ with $x_0 = -2, -1, 0$.

Exercise 2:

a) Determine the physically reasonable weak solution $u(x,t)$ to the Burgers' equation $u_t + uu_x = 0$ for the initial values

$$u(x,0) = \begin{cases} 0 & x \le 0, \\ 1 & 0 < x \le 1, \\ 2 & 1 < x. \end{cases}$$

b) Given the following initial value problem for \$u(x,t)\$:

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$
$$u(x,0) = \begin{cases} 2 & x \le 0, \\ 1 & 0 < x \le 2, \\ 0 & 2 < x. \end{cases}$$

(i) Compute the weak solution for $t \in [0, \tilde{t}\,]$ with a sufficiently small $\tilde{t}$.

(ii) To what maximum t^* can the solution from i) be continued?

(iii) Provide the weak solution for $t > t^*$.

Exercise 3:

Given a conservation equation $u_t + \left(\frac{u^4}{16}\right)_x = 0, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+ .$

a) Are the characteristics $(x(t), t)$ straight lines? Explain your answer.

b) Given the initial data $u(x,0) = 2 + \arctan(x)$, $x \in \mathbb{R}$. Determine the characteristic through the point $(0,0)$.

c) Check which of the functions $u^*, \tilde{u}, \hat{u}$ given below is the physically reasonable weak solution for the initial values

$$u(x,0) = \begin{cases} 2 & \text{for } x \le 0, \\ 1 & \text{for } x > 0 \end{cases}$$

$$u^*(x,t) = \begin{cases} 2 & \text{for } x \le \frac{3}{2}t, \\ 1 & \text{for } x > \frac{3}{2}t\,. \end{cases} \qquad \tilde{u}(x,t) = \begin{cases} 2 & \text{for } x \le \frac{15}{16}t, \\ 1 & \text{for } x > \frac{15}{16}t\,. \end{cases}$$

$$\hat{u}(x,t) = \begin{cases} 2 & \text{for } x \le 0, \\ 2 - \frac{x}{t} & \text{for } 0 < x \le t, \\ 1 & \text{for } x > t\,. \end{cases}$$

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