Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
Prof. Dr. T. Schmidt,
Dr. H. P. Kiani, J. Schütt

Differential Equations II for Engineering Students

Homework sheet 4

Exercise 1:

Determine physically reasonable weak solutions to the differential equation

$$u_t + (F(u))_x = 0$$

with the flow function $F(u) = \frac{(u-2)^4}{2}$ and initial conditions

a) $u(x,0) = \begin{cases} 2 & x \leq 0, \\ 1 & 0 < x, \end{cases}$ and **b)** $u(x,0) = \begin{cases} 1 & x \leq 0, \\ 2 & 0 < x. \end{cases}$

Note: Only solutions for the given initial values are required. You don't need to give solutions for general initial values!

Exercise 2:

Determine the physically reasonable weak solution to the Burgers' equation $u_t + uu_x = 0$ with the initial data

$$u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \leq x \leq 1 \\ 0 & x > 1 \end{cases}$$

at the time $t = 2$. What new problem occurs at $t = 2$?

Determine the solution for $t > 2$.

Exercise 3:

We discuss again the simple traffic flow model from Sheet 2 with the notation introduced there:

$u(x,t)$ = density of vehicles (vehicles/length) at point x at time t,

$v(x,t)$ = velocity at point x at time t,

$q(x,t) = v(x,t)u(x,t) =$ Flow at point x at time t = number of vehicles passing x at time t per time unit.

We improve our model from Sheet 2 by incorporating maximal density and a maximal velocity

u_{max} = maximal density of vehicles (bumper to bumper),

v_{max} = maximal velocity

This can be done, for example, as follows:

$$v(u(x,t)) = v_{max}\left(1 - \frac{u(x,t)}{u_{max}}\right)$$

a) Set up the continuity equation ($u_t + q_x = 0$).

b) Show again that the characteristics are straight lines and determine their slopes.

c) Determine a weak solution for

$$v_{max} = 1 \quad \text{(Here we have scaled appropriately!)}$$

$$u(x,0) = \begin{cases} u_l = \frac{u_{max}}{2} & x \leq 0 \quad \text{(normal trafic)}, \\ u_m = u_{max} & 0 < x \leq 1 \quad \text{(red traffic light/traffic jam)}, \\ u_r = 0 & x > 1 \quad \text{(empty street)} \end{cases}$$

and $t \in (0, 2)$.

d) For the Burgers equation, we only allowed shock waves in the case $u_l > u_r$ and inserted a rarefaction wave for $u_l < u_r$.

Are rarefaction waves possible for $u_l < u_r$?

Do shock waves only make physical sense for $u_l > u_r$?

Obviously other conditions must be met here. What could be the reason for this?

Note: A complete answer to the question is not possible with the help of the lecture slides alone. You can only make an assumption here.

Submission date: 03.06.-07.06.2024