Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
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Differential Equations II for Engineering Students

Work sheet 2

Exercise 1:

(a) Let $a \neq 0$ be a given constant and $u_0 : \mathbb{R} \to \mathbb{R}$ be a continuously differentiable function. Show that

$$u(t,x) = u_0(x - at),$$

is a solution of the initial value problem

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + a\dfrac{\partial u}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = u_0(x), & \text{if} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

(b) Find the solutions u and v of the following initial value problems:

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + \dfrac{\partial u}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = e^{-x^2}, & \text{if} \quad t = 0,\ x \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + \dfrac{\partial v}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ v(0,x) = \sin(\pi x), & \text{if} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

Show that $w = u + v$ is a solution of

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + \dfrac{\partial w}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) = e^{-x^2} + \sin(\pi x), & \text{if} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

Exercise 2:

(a) Show that

$$u(t,x) = \frac{x}{t+1} \qquad \text{and} \qquad v(t,x) = 1 \qquad \text{for} \quad (t,x) \in [0,\infty) \times \mathbb{R}$$

are solutions of the initial value problems

$$\begin{cases} \dfrac{\partial u}{\partial t}(t,x) + u(t,x)\dfrac{\partial u}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ u(0,x) = x, & \text{if} \quad t = 0,\ x \in \mathbb{R}, \end{cases}$$

and

$$\begin{cases} \dfrac{\partial v}{\partial t}(t,x) + v(t,x)\dfrac{\partial v}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ v(0,x) = 1, & \text{if} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

(b) Show that $w = u + v$ with u and v from part (a) is *not a* solution of

$$\begin{cases} \dfrac{\partial w}{\partial t}(t,x) + w(t,x)\dfrac{\partial w}{\partial x}(t,x) = 0, & \text{if} \quad (t,x) \in (0,\infty) \times \mathbb{R}, \\ w(0,x) = x + 1, & \text{if} \quad t = 0,\ x \in \mathbb{R}. \end{cases}$$

What is the main difference between the differential equation considered here and the differential equation in problem 1?

Exercise 3:

We consider the initial value problem

$$\begin{aligned} u_{tt} + u_{xt} - 2u_{xx} &= 0, \quad \text{if } x \in \mathbb{R},\ t \in \mathbb{R}^+, \\ u(x,0) &= \cos(x), \quad \text{if } x \in \mathbb{R}, \\ u_t(x,0) &= -4\sin(x), \quad \text{if } x \in \mathbb{R}. \end{aligned}$$

Solve the equation by using the substitution $\alpha = x + t,\ \mu = x - 2t$.

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