Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
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Differential Equations II for Engineering Students

Homework sheet 2

Exercise 1:

During this exercise we consider the differential equations for $u : D \to \mathbb{R}$, $D \subset \mathbb{R}^2$,

$$u_t(x,t) - \epsilon u_{xx}(x,t) = 0, \qquad \epsilon \in \mathbb{R}^+, \tag{1}$$

$$u_t(x,t) + \left(\frac{(u(x,t))^2}{2}\right)_x = 0, \tag{2}$$

$$u_t(x,t) + \left(\frac{(u(x,t))^2}{2}\right)_x - \epsilon u_{xx}(x,t) = 0, \tag{3}$$

$$(u_x(x,y))^2 - (u_y(x,y))^2 - u(x,y) = 0. \tag{4}$$

a) Specify the order of each of the equations and decide whether it is a linear, semilinear, quasilinear or (fully) non-linear equation.

b) Let $u^{[1]}$ and $u^{[2]}$ be two different, non-constant solutions of the above differential equations.

For the equations (1) to (4), check whether $\tilde{u} := k \cdot u^{[1]}$ is also a solution for any $k \in \mathbb{C}$ (or $\mathbb{R}$). If yes, check whether $\hat{u} := u^{[1]} + u^{[2]}$ is a solution of the differential equation, as well.

Note that in this case, every linear combination of (any number of) solutions of the differential equation is a solution of the considered differential equation (induction argument).

Exercise 2:

A simple traffic flow model:

We consider a one-dimensional flow of vehicles along an infinitely long, single-lane road. In a so-called macroscopic model, one does not consider individual vehicles, but the total flow of vehicles. For this purpose, we introduce the following quantities:

$$\begin{aligned} u(x,t) &= \text{(length-)density of the vehicles at the point } x \text{ at the time } t \\ &= \text{vehicles/unit length at point } x \text{ at the time } t \\ v(x,t) &= \text{speed at the point } x \text{ at the time } t \\ q(x,t) &= u(x,t)\cdot v(x,t) = \text{flow} \\ &= \text{amount of vehicles passing the point } x \text{ at the time } t \text{ per unit time} \end{aligned}$$

a) Assume that there are no entrances or exits, no vehicles are disappearing, and no new vehicles are appearing. Let $N(t,a,\Delta a) :=$ number of vehicles on a space interval $[a, a+\Delta a]$ at the time t.

Then on the one hand it holds that

$$N(t,a,\Delta a) = \int_a^{a+\Delta a} u(x,t)\,dx$$

and on the other hand it also holds

$$N(t,a,\Delta a) - N(t_0,a,\Delta a) = \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau)d\tau\,.$$

Derive from this the so-called conservation equation for the mass (number of vehicles)

$$u_t + q_x = 0.$$

Hints on how to proceed:

- Differentiate both formulas for N with respect to t. Please note that for the differentiation of parameter-dependent integrals with sufficiently smooth f holds the **Leibniz rule**:

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))}$$

- Divide by Δa.
- Consider the limit $\Delta a \to 0$.

b) We now assume in a first simple model that the speed increases in inverse proportion to the density and that the density is positive.

$$v(x,t) = c + \frac{k}{u(x,t)}$$

What is the continuity equation (=conservation equation for the mass)?

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