Fachbereich Mathematik der Universität Hamburg **SuSe 2024**
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Differential Equations II for Engineering Students

Homework sheet 1

Exercise 1: (Repetition Analysis II)

For the derivation of parameter-dependent integrals the **Leibniz rule**

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))$$

applies if f is sufficiently smooth.

a) Find the derivative of the function $F(x)$

$$F(x) := \int_{-x}^{x^2} e^{xt}\,dt$$

(i) by first integrating with respect to t and then deriving with respect to x,

(ii) by first deriving with respect to x and then integrating with respect to t.

b) Compute $\lim\limits_{x\to 0} F'(x)$.

Exercise 2:

The purpose of this exercise is to repeat the *differential operators*

$$\text{div}, \quad \textbf{grad}, \quad \textbf{rot}, \quad \Delta, \quad \nabla$$

which are known from Analysis III.

Let $D \subset \mathbb{R}^3$ be an open set and $\boldsymbol{x} = (x_1, x_2, x_3)^\top \in D$. We consider the functions

- $\boldsymbol{f} : D \to \mathbb{R}^3$ mit $\boldsymbol{f}(\boldsymbol{x}) = (f_1(\boldsymbol{x}), f_2(\boldsymbol{x}), f_3(\boldsymbol{x}))^\top$,
- $g : D \to \mathbb{R}$,

where both $\boldsymbol{f}$ and g are $\mathcal{C}^3$-functions.

(a) Indicate which of the following expressions is defined. If it is defined, identify whether the corresponding expression is a vector in $\mathbb{R}^3$ or a number in $\mathbb{R}$.

(i) $\text{div}(\textbf{grad}\, f)(\boldsymbol{x})$,

(ii) $\textbf{grad}\,(\Delta g)(\boldsymbol{x})$,

(iii) $\textbf{rot}\,(\text{div}\,\boldsymbol{f})(\boldsymbol{x})$,

(iv) $\Delta\,(\text{div}\,\boldsymbol{f})(\boldsymbol{x})$.

(b) Show the two equalities

$$\operatorname{div}(\,\mathbf{rot}\, f)\,(\,\boldsymbol{x}\,) = 0 \qquad \text{and} \qquad \mathbf{rot}\,(\nabla g)\,(\,\boldsymbol{x}\,) = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

(c) Show the following equality

$$\nabla(\operatorname{div}\,\boldsymbol{f}\,)(\,\boldsymbol{x}\,) - \,\mathbf{rot}(\,\mathbf{rot}\,\boldsymbol{f}\,)(\,\boldsymbol{x}\,) = \begin{pmatrix} \Delta f_1(\boldsymbol{x}) \\ \Delta f_2(\boldsymbol{x}) \\ \Delta f_3(\boldsymbol{x}) \end{pmatrix}.$$

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