Mathematics Department
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WiSe 2024/25

Exam: Differential Equations II

04. March 2025

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Task no.	**Points**	**Evaluator**
1		
2		
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Exercise 1: [5 Points]

Given the following initial value problem

$$u_t + t \cdot u_x = 1, \qquad \text{for} \qquad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = \cos(x), \qquad \text{for} \qquad x \in \mathbb{R}.$$

a) state the characteristic equations for this problem and determine their solutions,

b) solve the initial value problem.

Solution:

a) With the characteristic method, we obtain for
$\gamma(t) = \binom{x(t)}{t}$ with $\dot{\gamma}(t) = \binom{\dot{x}(t)}{1}$
and $\nu(t) := u(\gamma(t))$
the characteristic equations

$$\dot{x}(t) = t,\ x(0) = x_0, \qquad \text{und} \qquad \dot{\nu}(t) = 1, \quad \nu(0) = \nu_0.$$

From the first equation we obtain

$$x(t) = \frac{t^2}{2} + x_0.$$

Therefore, we obtain with the second equation.

$$\nu(t) = t + \nu_0$$

(b) From $x = \frac{t^2}{2} + x_0$ it follows that

$$x_0 = x - \frac{t^2}{2}$$

and with

$$\nu_0 = \nu(0) = u(x_0, 0) = \cos(x_0),$$

we obtain

$$u(x,t) = t + \cos\left(x - \frac{t^2}{2}\right).$$

Exercise 2: [6 Points]

For $u(x,t)$ the following initial value problem is given:

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} 2 & x \leq -2, \\ 0 & -2 < x \leq 1, \\ -1 & 1 < x\,. \end{cases}$$

a) Determine the physically reasonable solution of the initial value problem for $t \in [0, t^*)$ with suitably small t^*.

b) Up to which t^* does the solution formula from a) make sense?

c) How can the solution be extended for $t > t*$ in a physically reasonable way?

Solution:

a) We insert shock waves at the two jump points of the initial data.

The jump condition requires:

$\dot{s}_1(t) = \dfrac{2+0}{2} = 1$ and $\dot{s}_2(t) = \dfrac{0-1}{2} = -\dfrac{1}{2}$.

We obtain the shock waves

$s_1(t) = -2 + t$ and $s_2(t) = 1 - \dfrac{t}{2}$.

For suitably small t the function

$$u(x,t) = \begin{cases} 2 & x \leq -2+t, \\ 0 & -2+t < x \leq 1 - \frac{t}{2}, \\ -1 & 1 - \frac{t}{2} < x\,. \end{cases} \qquad \textbf{(3 Points)}$$

is a weak solution.

b) For t^* with

$$-2 + t^* = 1 - \frac{t^*}{2} \iff -4 + 2t^* = 2 - t^* \iff t^* = 2 \qquad \textbf{(1 Point)}$$

the two shock waves meet and the solution of a) becomes ambiguous.

c) For $t^* = 2$ it holds $s_1(2) = s_2(2) = 0$ and

$$u(x,2) = \begin{cases} 2 & x \leq 0\,, \\ -1 & x > 0\,. \end{cases}$$

We insert a new shock font s_3 with $\dot{s}_3(t) = \frac{2+(-1)}{2} = \frac{1}{2}$.

$$s_3(t) = s_3(2) + \dot{s}_3(t)(t-2) = 0 + \frac{t-2}{2}$$

For $t > 2$ we obtain

$$u(x,t) = \begin{cases} 2 & x \leq \frac{t-2}{2}, \\ -1 & x > \frac{t-2}{2}. \end{cases} \qquad \textbf{(2 Points)}$$

Exercise 3: [6 Points],

Determine the solution of the following initial boundary value problem:

$$\begin{aligned}
u_{tt} - 9u_{xx} &= 0 && 0 < x < 2\,,\ 0 < t\,,\\
u(x,0) &= 5\sin(2\pi x) + 7\sin(3\pi x) && 0 \le x \le 2,\\
u_t(x,0) &= 9\sin(\pi x) && 0 \le x \le 2,\\
u(0,t) &= 0 && 0 \le t,\\
u(2,t) &= 0 && 0 \le t.
\end{aligned}$$

Solution:

With $L = 2$ and $c = +\sqrt{9}$ the solution formula reads:

$$u(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{ck\pi}{L}t\right) + B_k \sin\left(\frac{ck\pi}{L}t\right)\right] \sin\left(\frac{k\pi}{L}x\right).$$

Thus,

$$u(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{3k\pi}{2}t\right) + B_k \sin\left(\frac{3k\pi}{2}t\right)\right] \sin\left(\frac{k\pi}{2}x\right). \qquad \textbf{(1 Point)}$$

Therefore, for $t = 0$

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k\pi}{2}x\right) \stackrel{!}{=} 5\sin(2\pi x) + 7\sin(3\pi x).$$

Thus, $A_4 = 5$, $A_6 = 7$ and $A_k = 0$ else. **(2 Points)**

It holds

$$u_t(x,t) = \sum_{k=1}^{\infty} \left[-A_k \cdot \frac{3k\pi}{2} \cdot \sin\left(\frac{3k\pi}{2}t\right) + B_k \cdot \frac{3k\pi}{2} \cdot \cos\left(\frac{3k\pi}{2}t\right)\right] \sin\left(\frac{k\pi}{2}x\right)$$

and for $t = 0$:

$$u_t(x,0) = \sum_{k=1}^{\infty} \frac{3k\pi}{2} B_k \sin\left(\frac{k\pi}{2}x\right) \stackrel{!}{=} 9\sin(\pi x).$$

Comparison of coefficients yields:

$B_k = 0,\ \forall k \neq 2\,,$

$3\pi \cdot B_2 \stackrel{!}{=} 9 \implies B_2 = \frac{3}{\pi}$ else. **(2 Points)**

The solution is therefore

$$\begin{aligned}
u(x,t) &= A_4 \cos\left(\frac{12\pi}{2}t\right)\sin\left(\frac{4\pi}{2}x\right) + A_6 \cos\left(\frac{18\pi}{2}t\right)\sin\left(\frac{6\pi}{2}x\right) + B_2 \sin\left(\frac{6\pi}{2}t\right)\sin\left(\frac{2\pi}{2}x\right)\\
&= 5\cos(6\pi t)\sin(2\pi x) + 7\cos(9\pi t)\sin(3\pi x) + \frac{3}{\pi}\sin(3\pi t)\sin(\pi x). \qquad \textbf{(1 Point)}
\end{aligned}$$

Exercise 4: [3 Points]

Let $\tilde{u}$ and $\hat{u}$ be solutions of the differential equation

$$u_t - u_{xx} + u = 2, \qquad x \in (0,1),\, t \in \mathbb{R}^+,$$

for $u(x,t)$, which satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = \sin(t), \qquad t \in \mathbb{R}^+.$$

a) Is $\tilde{u} + \hat{u}$ a solution of the differential equation?
Justify your answer.

b) Does $\tilde{u} - \hat{u}$ solve the following differential equation

$$u_t - u_{xx} + u = 0, \qquad x \in (0,1),\, t \in \mathbb{R}^+,$$

and satisfy the boundary conditions

$$u(0,t) = 0, \qquad u(1,t) = \sin(t), \qquad t \in \mathbb{R}^+?$$

Solution:

a) No. The differential equation is linear but not homogeneous. For $v := \tilde{u}+\hat{u}$ one obtains

$$\begin{aligned} v_t - v_{xx} + v &= (\tilde{u}_t + \hat{u}_t) - (\tilde{u}_{xx} + \hat{u}_{xx}) + \tilde{u} + \hat{u} \\ &= (\tilde{u}_t - \tilde{u}_{xx} + \tilde{u}) + (\hat{u}_t - \hat{u}_{xx} + \hat{u}) = 2 + 2 \neq 2. \end{aligned}$$

b) No. $\tilde{u} - \hat{u}$ satisfies the homogeneous differential equation (same computation as in a)) but it does not satisfy the boundary conditions:

$$(\tilde{u} - \hat{u})(1,t) = \tilde{u}(1,t) - \hat{u}(1,t) = \sin(t) - \sin(t) = 0.$$