Mathematics Department
Prof. Dr. T. Schmidt

SuSe 2024

Exam Differential Equations II

26. August 2024

Please mark each page with your name and your matriculation number.

Please write your surname, first name and matriculation number each in block letters in the designated fields following. These entries will be stored on data carriers.

Surname:																
First name:																
Matr.-No.:																

Bachelor's Program:	CI	CS	ES	GES	

I was instructed about the fact that the exam performance will only be assessed if the Central Examination Office of TUHH verifies my official admission before the exams beginning in retrospect.

Signature:

Task no.	Points	Evaluator
1		
2		
3		
4		

$\sum =$

Exercise 1: [5 Points]

Given the following initial value problem

$$u_t + \frac{1}{t+1} \cdot u_x = u \qquad \text{for} \qquad x \in \mathbb{R},\ t > 0,$$

$$u(x,0) = e^{-x} \qquad \text{for} \qquad x \in \mathbb{R},$$

a) state the characteristic equations for this problem and determine their solutions,

b) solve the initial value problem.

Solution:

a) With the characteristic method, we compute:
$\gamma(t) = \binom{x(t)}{t}$ with $\dot\gamma(t) = \binom{\dot x(t)}{1}$
and $\nu(t) := u(\gamma(t))$
$\frac{dx}{dt} = \frac{1}{t+1} \Longrightarrow x(t) = \ln(t+1) + C_1.$
From $x(0) = \ln(1) + C_1 = C_1$ it follows that
$x(t) = \ln(t+1) + x(0)$.
$\frac{d\nu}{dt} = \nu(t) \Longrightarrow \nu(t) = C_2 e^t.$
From $\nu(0) = C_2 e^0 = C_2$ we obtain
$\nu(t) = \nu(0)e^t.$ **[3 Points]**

b) From $x = \ln(t+1) + x(0)$ it follows that

$$x(0) = x(t) - \ln(t+1)$$

and with

$$\nu_0 = \nu(0) = u(x_0, 0) = e^{-x_0}$$

we obtain

$$u(x,t) = e^{-(x-\ln(t+1))} \cdot e^t = (t+1)e^{t-x}.$$

[2 Points]

Exercise 2: [4+1 Points]

Given the following initial value problem for $u(x,t)$

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} 2 & \text{for } x \leq -1, \\ 0 & \text{for } -1 < x \leq 0, \\ 1 & \text{for } 0 < x, \end{cases}$$

a) determine the physically reasonable solution of the initial value problem for $0 < t < 1$.

b) Why does the solution formula from a) only hold for $t < 1$?

Solution:

a) The solution is composed of the solutions of two Riemann problems. We denote the flow of the Burgers' equation by $F(u) = \frac{u^2}{2}$ and since $2 > 0$, we first obtain a shock wave $s(t)$ with

$$\dot{s}(t) = \frac{F(2) - F(0)}{2 - 0} = \frac{1}{2}(2 + 0) = 1 \quad \text{and} \quad s(0) = -1$$

$$\Rightarrow \quad s(t) = -1 + t. \qquad \textbf{[2 Points]}$$

Because of $0 < 1$, we obtain a rarefaction wave with boundaries

$$F'(0)t = 0, \qquad F'(1)t = t. \qquad \textbf{[1 Point]}$$

Inside the rarefaction wave u has the form

$$u(x,t) = (F')^{-1}\left(\frac{x}{t}\right) = \frac{x}{t}.$$

Thus together we have

$$u(x,t) = \begin{cases} 2 & \text{for} \quad x \leq -1 + t, \\ 0 & \text{for} \quad -1 + t < x < 0, \\ \frac{x}{t} & \text{for} \quad 0 \leq x \leq t, \\ 1 & \text{for} \quad t < x. \end{cases} \quad \textbf{[1 Point]}$$

b) At time point $t = 1$ the shock wave meets the rarefaction wave such that the solution formula from a) no longer holds. **[1 Point]**

Exercise 3: [1+2,5+2,5 Points]

Determine the bounded solution of the following boundary value problems for the Laplace equations: You can give the solutions in cartesian or polar coordinates.

a) $$\begin{cases} \Delta u = 0 & \text{on} \quad \Omega_1 := \{\binom{x}{y} \in \mathbb{R}^2,\, x^2+y^2 < 25\}, \\ u(x,y) = 4 & \text{for} \quad x^2+y^2 = 25. \end{cases}$$

b) $$\begin{cases} \Delta u = 0 & \text{on} \quad \Omega_1 := \{\binom{x}{y} \in \mathbb{R}^2,\, x^2+y^2 < 25\}, \\ u(x,y) = u(r\cos(\phi), r\sin(\phi)) = 3\sin(2\phi) & \text{for} \quad x^2+y^2 = 25. \end{cases}$$

c) $$\begin{cases} \Delta u = 0 & \text{on} \quad \Omega_2 := \{\binom{x}{y} \in \mathbb{R}^2,\, 1 < x^2+y^2 < 25\} \\ u(x,y) = 4 & \text{for} \quad x^2+y^2 = 1, \\ u(x,y) = 2 & \text{for} \quad x^2+y^2 = 25. \end{cases}$$

Solution: [1+2,5+2,5 Points]

a) The constant function $u(x,y) = 4$ solves the Laplace equation and in therefore the unique solution.

b) With $x = r\cos(\phi)$, $y = r\sin(\phi)$ and $v(r,\phi) = u(r\cos(\phi), r\sin(\phi))$ the representation of the solution is

$$v(r,\varphi) = a_0 + \sum_{k=1}^{\infty} \left(c_k\cos(k\varphi) + d_k\sin(k\varphi)\right) r^k .$$

The boundary values give the condition

$$\begin{aligned} v(5,\varphi) &= a_0 + \sum_{k=1}^{\infty} \left(c_k\cos(k\varphi) + d_k\sin(k\varphi)\right) 5^k \\ &= 3\sin(2\varphi) . \end{aligned}$$

Comparison of coefficients gives $25 d_2 \overset{!}{=} 3$ and that every other coefficient is 0.

Thus, we obtain the solution

$$v(r,\phi) = \frac{3r^2}{25}\sin(2\phi) .$$

A representation with respect to the cartesian coordinates is not required. If someone is doing it anyway, they should obtain

$$u(x,y) = \frac{6(x^2+y^2)}{25} \cdot \frac{x}{\sqrt{x^2+y^2}} \cdot \frac{y}{\sqrt{x^2+y^2)}} = \frac{6xy}{25}.$$

c) Since $(0,0)^\top \notin \Omega_2$, it holds with the fundamental solution $\Phi(x,y) = \frac{1}{2\pi}\ln(|(x,y)|) = \frac{1}{2\pi}\ln(\sqrt{x^2+y^2})$ that

$$u(x,y) = a\Phi(x,y) + b.$$

From the boundary values we obtain

$\frac{a}{2\pi}\ln(1) + b = b \overset{!}{=} 4$

$\frac{a}{2\pi}\ln(5) + 4 \overset{!}{=} 2 \;\Rightarrow\; a = \frac{-4\pi}{\ln(5)}$,

thus, $u(x,y) = 4 - \frac{4\pi}{\ln(5)} \cdot \frac{1}{2\pi}\ln(\sqrt{x^2+y^2}) = 4 - \frac{2}{\ln(5)}\ln(\sqrt{x^2+y^2})$.

Exercise 4: [3+1 Points]

Given the following initial boundary value problem

$$\begin{aligned} u_t - 16u_{xx} &= 4\cos(t)\Big(1 - \frac{x}{2\pi}\Big) && \text{for } x \in (0, 2\pi), t > 0, \\ u(x,0) &= \frac{x}{2\pi} && \text{for } x \in [0, 2\pi], \\ u(0,t) &= 4\sin(t), \quad u(2\pi, t) = 1 && \text{for } t > 0. \end{aligned}$$

a) transfer this problem into an initial boundary value problem with homogeneous boundary data for a function $v(x,t)$ via a suitable homogenization.

   Write down the new initial boundary value problem (differential equartion, initial values, boundary values).

b) Write down a solution v for the initial boundary value problem with homegeneous boundary data from part a) without any computations. What is the corresponding solution u of the originial problem?

Solution:

a) Homogenization:

   $v(x,t) = u(x,t) - \Big[4\sin(t) + \frac{x}{2\pi}(1 - 4\sin(t))\Big] = u(x,t) - \frac{x}{2\pi} + 4\sin(t)(\frac{x}{2\pi} - 1)\,.$

   or

   $u(x,t) = v(x,t) + \frac{x}{2\pi} - 4\sin(t)(\frac{x}{2\pi} - 1)\,.$ **[1 Point]**

   Then it holds:

   $u_t = v_t - 4\cos(t)(\frac{x}{2\pi} - 1),$

   New differential equation:

   $$v_t + 4\cos(t)(1 - \tfrac{x}{2\pi}) - 16v_{xx} = 4\cos(t)(1 - \tfrac{x}{2\pi}) \iff \boxed{v_t - 16v_{xx} = 0.}$$

   Initial values:

   $v(x,0) = u(x,0) - \frac{x}{2\pi} + 4\sin(0)(\frac{x}{2\pi} - 1) = \frac{x}{2\pi} - \frac{x}{2\pi} = 0.$ **[2 Points]**

   Boundary values:

   $v(0,t) = u(0,t) - \Big[4\sin(t) + \frac{0}{2\pi}(1 - 4\sin(t))\Big] = 4\sin(t) - 4\sin(t) = 0.$

   $v(2\pi,t) = u(2\pi,t) - \Big[4\sin(t) + \frac{2\pi}{2\pi}(1 - 4\sin(t))\Big] = 1 - [4\sin(t) + 1 - 4\sin(t)] = 0.$

b) Since the differential equation is homogeneous with vanishing initial and boundary dara, $v \equiv 0$ is the solution. Hence,

   $u(x,t) = 0 + \frac{x}{2\pi} - 4\sin(t)(\frac{x}{2\pi} - 1)\,.$ **[1 Point]**