Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Differential Equations II for Engineering Students

Work sheet 7

Exercise 1:

a) Given the initial boundary value problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= -x \cdot \sin(t) && \text{for } x \in (0,1), t > 0, \\ u(x,0) &= 1 - x + 4\sin(2\pi x) && \text{for } x \in [0,1], \\ u_t(x,0) &= x + 3\sin(6\pi x) && \text{for } x \in [0,1], \\ u(0,t) &= 1, \quad u(1,t) = \sin(t) && \text{for } t > 0. \end{aligned}$$

Transform the problem using suitable homogenization of the boundary data into an initial boundary value problem with homogeneous boundary data.

b) Solve the following initial boundary value problem

$$\begin{aligned} v_{tt} - 4v_{xx} &= 0 && \text{for } x \in (0,1), t > 0, \\ v(x,0) &= 4\sin(2\pi x) && \text{for } x \in [0,1], \\ v_t(x,0) &= 3\sin(6\pi x) && \text{for } x \in [0,1], \\ v(0,t) &= 0, \quad v(1,t) = 0 && \text{for } t > 0. \end{aligned}$$

Solution sketch:

a) Homogenization:

$$v(x,t) = u(x,t) - 1 - \frac{x}{L}(\sin(t) - 1) = u(x,t) - 1 - x\sin(t) + x\,.$$

or

$$u(x,t) = v(x,t) + 1 + x\sin(t) - x. \qquad \textbf{[1 point]}$$

Then it holds:

$$u_t = v_t + x\cos(t), \; u_x = v_x + \sin(t) - 1$$
$$u_{tt} = v_{tt} - x\sin(t), \; v_{xx} = u_{xx} \qquad \textbf{[1 point]}$$

New differential equation:

$$v_{tt} - x\sin(t) - 4v_{xx} = -x \cdot \sin(t) \iff \boxed{v_{tt} - 4v_{xx} = 0\,.} \qquad \textbf{[1 point]}$$

Initial data:

$$v(x,0) = u(x,0) - 1 - x(\sin(0) - 1) = 1 - x + 4\sin(2\pi x) - 1 + x \Longrightarrow$$

$$\boxed{v(x,0) = 4\sin(2\pi x)} \qquad \textbf{[1 point]}$$

$$v_t(x,0) = u_t(x,0) - x\cos(0) = x + 3\sin(6\pi x) - x \Longrightarrow$$

$$\boxed{v_t(x,0) = 3\sin(6\pi x)} \qquad \textbf{[1 point]}$$

Boundary data : $\boxed{v(0,t) = v(1,t) = 0}$

b) With $L = 1$ and $c^2 = 4$ we have a solution formula:

$$v(x,t) \;=\; \sum_{k=1}^{\infty} \left[A_k \, \cos\left(\frac{ck\pi}{L} t\right) \;+\; B_k \, \sin\left(\frac{ck\pi}{L} t\right)\right] \, \sin\left(\frac{k\pi}{L} x\right)$$

So for $t = 0$ we have

$$v(x,0) \;=\; \sum_{k=1}^{\infty} A_k \, \sin\left(k\pi x\right) \overset{!}{=} 4\sin(2\pi x)$$

Also $A_2 = \; 4$ and $A_k = 0$ else. **[2 points]**

$$v_t(x,t) \;=\; \sum_{k=1}^{\infty} \left[-A_k \cdot 2k\pi \cdot \sin\left(2k\pi t\right) \;+\; B_k \;\cdot 2k\pi \cdot \cos\left(2k\pi t\right)\right] \, \sin\left(k\pi x\right)$$

and for $t = 0$:

$$v_t(x,t) \;=\; \sum_{k=1}^{\infty} B_k \;\cdot 2k\pi \, \sin\left(k\pi x\right) \overset{!}{=} 3\sin(6\pi x)$$

Also $B_6 = \frac{3}{2 \cdot 6 \cdot \pi} \; = \frac{1}{4\pi}$ and $B_k = 0$ else.

$$v(x,t) \;=\; 4\, \cos\left(4\pi t\right) \, \sin\left(2\pi x\right) \;+\; \frac{1}{4\pi} \, \sin\left(12\pi t\right) \, \sin\left(6\pi x\right)$$ **[2 points]**

Exercise 2:

From lecture classes you know d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}\,\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha$$

for the solution of the initial value problem for the (homogeneous) wave equation

$$\hat{u}_{tt} - c^2\hat{u}_{xx} = 0,\ \hat{u}(x,0) = f(x),\ \hat{u}_t(x,0) = g(x),\ x \in \mathbb{R},\ c > 0.$$

a) **(Just for the really quick participants)** Show that the function

$$\tilde{u}(x,t) = \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau$$

solves the following inhomogeneous initial value problem.

$$\tilde{u}_{tt} - c^2\tilde{u}_{xx} = h(x,t) \qquad \tilde{u}(x,0) = \tilde{u}_t(x,0) = 0.$$

Hint: Leibniz formula for the derivation of parameter-dependent integrals (Sheet 1H):

$$\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))$$

b) Solve the initial value problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= -4x, & x \in \mathbb{R},\ t > 0 \\ u(x,0) &= 1, & x \in \mathbb{R}, \\ u_t(x,0) &= \cos(x), & x \in \mathbb{R} \end{aligned} \tag{1}$$

(i) Compute a solution $\hat{u}$ of the initial value problem

$$\begin{aligned} \hat{u}_{tt} - 4\hat{u}_{xx} &= 0, & x \in \mathbb{R},\ t > 0 \\ \hat{u}(x,0) &= 1,\ x \in \mathbb{R}, \\ \hat{u}_t(x,0) &= \cos(x),\ x \in \mathbb{R}. \end{aligned}$$

(ii) Compute a solution $\tilde{u}$ of the initial value problem using the result from part a).

$$\begin{aligned} \tilde{u}_{tt} - 4\tilde{u}_{xx} &= -4x, & x \in \mathbb{R},\ t > 0 \\ \tilde{u}(x,0) &= 0,\ x \in \mathbb{R}, & \tilde{u}_t(x,0) = 0,\ x \in \mathbb{R} \end{aligned}$$

(iii) By inserting u into the differential equation and checking the initial values, show that $u = \tilde{u} + \hat{u}$ solves the initial value problem (1).

Solution:

a) $$\tilde{u}_x(x,t) = \frac{1}{2c}\int_0^t [h(x-c(\tau-t),\tau) - h(x+c(\tau-t),\tau]\,\mathrm{d}\tau$$

$$\tilde{u}_{xx}(x,t) = \frac{1}{2c}\int_0^t [h_\omega(x-c(\tau-t),\tau) - h_\omega(x+c(\tau-t),\tau]\,\mathrm{d}\tau$$

$$\begin{aligned}\tilde{u}_t(x,t) &= \frac{1}{2c}\int_{x+c(t-t)}^{x-c(t-t)} h(\omega,t)\,\mathrm{d}\omega \\ &+ \frac{1}{2c}\int_0^t [h(x-c(\tau-t),\tau)\cdot c - h(x+c(\tau-t),\tau)\cdot(-c)]\,\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^t [h(x-c(\tau-t),\tau) + h(x+c(\tau-t),\tau)]\,\mathrm{d}\tau\end{aligned}$$

$$\begin{aligned}\tilde{u}_{tt}(x,t) &= \frac{1}{2}\left\{h(x,t) + h(x,t) + \int_0^t [h_\omega(x-c(\tau-t),\tau)\cdot c + h_\omega(x+c(\tau-t),\tau)(-c)]\,\mathrm{d}\tau\right\} \\ &= h(x,t) + \frac{c}{2}\int_0^t [h_\omega(x-c(\tau-t),\tau) - h_\omega(x+c(\tau-t),\tau)]\,\mathrm{d}\tau\end{aligned}$$

Obviously it holds $\tilde{u}_{tt} - c^2\tilde{u}_{xx} = h(x,t)$. For initial values one obtains

$$\tilde{u}(x,0) = \frac{1}{2c}\int_0^0 \cdots = 0\,,$$

and

$$\tilde{u}_t(x,0) = \frac{1}{2}\int_0^0 \cdots = 0\,.$$

b) (i) Solution to a homogeneous differential equation with inhomogeneous initial values according to d'Alembert

$$\begin{aligned}\hat{u}(x,t) &= \frac{1}{2}(1+1) + \frac{1}{2c}\int_{x-ct}^{x+ct}\cos(\eta)d\eta \\ &= 1 + \frac{1}{4}(\sin(x+2t) - \sin(x-2t)) \\ &= 1 + \frac{1}{2}\cos(x)\sin(2t)\end{aligned}$$

(ii) Solution of a inhomogeneous differential equation with homogeneous initial values

$$\begin{aligned}\tilde{u}(x,t) &= \frac{1}{2c}\int_0^t\int_{x+c(\tau-t)}^{x-c(\tau-t)} -4\omega\,\mathrm{d}\omega\mathrm{d}\tau = \frac{-4}{8}\int_0^t \left[(x-2(\tau-t))^2 - (x+2(\tau-t))^2\right]\mathrm{d}\tau \\ &= \frac{1}{2}\int_0^t 8x(\tau-t)\mathrm{d}\tau = -2xt^2.\end{aligned}$$

(iii) The solution to the original problem consists of the two partial solutions:

$$u(x,t) = 1 + \frac{1}{2}\cos(x)\sin(2t) - 2xt^2$$

Test:

$u(x,0) = 1,\ u_t(x,t) = \cos(x)\cos(2t) - 4xt,\ u_t(x,0) = \cos(x),$

$u_x = -\frac{1}{2}\,\sin(x)\sin(2t)\ - 2t^2,\ u_{xx} = -\dfrac{1}{2}\,\cos(x)\sin(2t),$

$u_{tt} = -2\cos(x)\sin(2t) - 4x\,.$

$u_{tt} - 4u_{xx} = -2\cos(x)\sin(2t) - 4x - 4(-\dfrac{1}{2}\,\cos(x)\sin(2t)) = -4x\,.$

Discussion: 10.07 - 14.07.2023