Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Differential Equations II for Engineering Students

Homework sheet 7

Exercise 1: Solve the initial boundary value problem:

$$\begin{aligned}
u_{tt} - 4u_{xx} &= 3\sin(2\pi x)\cdot e^{-2t} \qquad x\in(0,1),\ t>0\\
u(x,0) &= u_0(x) = \sin(\pi x) + 4\sin(2\pi x) \qquad x\in[0,1],\\
u_t(x,0) &= v_0(x) = 0 \qquad x\in[0,1],\\
u(0,t) &= 0 \qquad t>0,\\
u(1,t) &= 0 \qquad t>0,
\end{aligned}$$

Hint: Insert the ansatz

$$u(x,t) = \sum_{k=1}^{\infty} q_k(t)\sin(k\omega x), \qquad \omega = \frac{\pi}{1}$$

into the differential equation. You get the ordinary differential equation for q_k. The initial conditions provide the initial data for the q_k.

Solution:

Inserting the ansatz in $\omega = \pi$ and $c = 2$

$$\sum_{k=1}^{\infty} q_k''(t)\sin(k\omega x) + c^2(k\omega)^2 q_k(t)\sin(k\omega x) \overset{!}{=} 3\sin(2\pi x)\cdot e^{-2t}$$

We obtain the ordinary differential equation

$$q_2''(t) + 2^2(2\pi)^2 q_2(t) = 3e^{-2t}$$
$$q_k''(t) + 2^2(k\pi)^2 q_k(t) = 0 \qquad \forall\, k \neq 0$$

We now insert the ansatz into the initial conditions

$$u_0(x) = \sum_{k=1}^{\infty} q_k(0)\sin(k\pi x) \overset{!}{=} \sin(\pi x) + 4\sin(2\pi x)$$

So we obtain from $q_1(0) = 1$, $q_2(0) = 4$, $q_k(0) = 0$ else!

The second initial condition together with the solution ansatz gives

$$v_0(x) = \sum_{k=1}^{\infty} q_k'(0)\sin(k\omega x) \overset{!}{=} 0$$

Hence $\quad q_k'(0) = 0 \;\forall k \in \mathbb{N}\,.$

For $k \notin \{1,2\}$ we have the initial value problem

$$q_k''(t) + c^2k^2\omega^2 q_k(t) \;=\; 0\,, \qquad q_k(0) = 0,\; q_k'(0) = 0\,.$$

With the solution : $q_k(t) = 0\,.$

For $k = 1$ we have with $c = 2$ and $\omega = \pi$ the initial value probleem:

$$q_1''(t) + 4\pi^2 q_1(t) \;=\; 0\,, \qquad q_1(0) = 1,\; q_1'(0) = 0\,.$$

With the general solution: $\qquad q_1(t) = \; k_1\cos(2\pi t) + \hat{k}_1\sin(2\pi t)\,.$

Adjusting the initial values we have $q_1(t) = \; \cos(2\pi t)$

For $k = 2$ we have the initial value problem:

$$q_2''(t) + 16\cdot\pi^2 q_2(t) \;=\; 3e^{-2t}\,, \qquad q_2(0) = 4\,,\; q_2'(0) = 0$$

Corresponding homogeneous differential equation: $q_{2,h}''(t) + 4^2\cdot\pi^2 q_{2,h}(t) \;=\; 0,$

with the general solution: $q_{2,h}(t) \;=\; k_2\cos(4\pi t) + \hat{k}_2\sin(4\pi t)$

Ansatz for the concrete solution of the inhomogeneous differential equation: $q_{2,p}(t) \;=\; a\,e^{-2t}$

$$4a + 16\cdot\pi^2 a \;=\; 3 \implies a = \frac{3}{4+16\pi^2}$$

$$q_2(t) \;=\; k_2\cos(4\pi t) + \hat{k}_2\sin(4\pi t) \;+\; \frac{3}{4+16\pi^2}\,e^{-2t}$$

$$q_2(0) = k_2 \;+\; \frac{3}{4+16\pi^2} \;=\; 4 \iff k_2 \;=\; 4 - \frac{3}{4+16\pi^2}$$

$$q_2'(0) = \hat{k}_2\,4\pi \;-\; \frac{6}{4+16\pi^2} \;=\; 0 \iff \hat{k}_2 \;=\; \frac{3}{2\pi(4+16\pi^2)}$$

$$q_2(t) \;=\; \frac{13+64\pi^2}{4+16\pi^2}\cos(4\pi t) + \frac{3}{2\pi(4+16\pi^2)}\sin(4\pi t) \;+\; \frac{3}{4+16\pi^2}\,e^{-2t}$$

And then we have

$$u(x,t) = \; \cos(2\pi t)\sin(1\cdot\pi x)$$
$$+ \; \frac{1}{4+16\pi^2}\left((13+64\pi^2)\,\cos(4\pi t) + \frac{3}{2\pi}\sin(4\pi t) \;+\; 3\,e^{-2t}\right)\sin(2\cdot\pi x)\,.$$

Exercise 2: *(So that you don't get the idea that you can solve all linear differential equations of second order with a simple product ansatz.)*

At the starting point $x = 0$ of a very long transmission cable there is a signal of the periodic voltage

$$U(0,t) \;=\; U_0 \cos(\omega t) \qquad t \geq 0,\, \omega > 0\,.$$

We are looking for the signal voltage $U(x,t)$ of the output signal for $x > 0, t > 0$. One obtains U as the solution of the so-called telegrapher's equations.

$$U_{tt} \;-\; c^2 U_{xx} \;+\; (\alpha+\beta)U_t \;+\; \alpha\beta U \;=\; 0\,.$$

Here α, β, c are construction-related positive parameters of the problem. A temporally periodic input signal leads to expect a temporally periodic output signal after a certain settling phase. In addition, one requires

$$U(x,t) \quad \text{bounded for} \quad x \to \infty\,.$$

a) Show that the product ansatz $U(x,t) = w(x) \cdot v(t)$ is not working here!

b) Try the solution ansatz that combines "local damping"(factor e^{-kx}) with a time-periodic progression (i.e. cosine/sine for t) and allows a linear, location-dependent phase shift. So for example

$$U(x,t) := e^{-kx} \cdot \Big(\delta \cos(at - bx) \;+\; \tilde{\delta} \sin(\tilde{a}t - \tilde{b}x)\Big)$$

Choose for example $\alpha = \beta = c = 1$.

Solution to Exercise 2:

a) A product ansatz of the form $U(x,t) \;=\; v(t)w(x)$ leads to the ordinary differential equation for v

$$\frac{\ddot{v}(t) + (\alpha+\beta)\dot{v}(t) + \alpha\beta v(t)}{v(t)} \;=\; c^2 \frac{w''(x)}{w(x)} \;=:\; K.$$

We have

$$\ddot{v}(t) + (\alpha+\beta)\dot{v}(t) + \alpha\beta v(t) = \; K \cdot v(t) \iff \ddot{v} + (\alpha+\beta)\dot{v} + (\alpha\beta - K)v = 0$$

This is an ordinary linear differential equation with constant coefficients for v. So we compute the zeros of the characteristic polynomial

$$\mu^2 + (\alpha+\beta)\mu + (\alpha\beta + \lambda) = 0 \iff \mu_{1,2} = -\frac{\alpha+\beta}{2} + \sqrt{\frac{(\alpha+\beta)^2}{4} \;-\; (\alpha\beta - K)}$$

The general solution has the form

$v(t) = c_1 e^{\mu_1 t} + c_2 e^{\mu_2 t}$ and $v(t) = c_1 e^{\mu_1 t} + c_2 t e^{\mu_1 t}$.

This is periodic if and only if $\mu_1 = \overline{\mu_2}$ are purely imaginary. The latter is only possible if $\alpha + \beta = 0$ applies. But α and β are positive constants according to the problem. So our product ansatz does not lead to the solution.

b) Ansatz: $U(x,t) := e^{-kx} \cdot \Big(\delta \cos(at - bx) + \tilde{\delta} \sin(\tilde{a}t - \tilde{b}x)\Big)$

$$U(0,t) = \delta \cos(at) + \tilde{\delta} \sin(\tilde{a}t) \overset{!}{=} U_0 \cos(\omega t) \implies$$

$\delta = U_0,\ \tilde{\delta} = 0,\ a = \omega$

It also holds

$$\begin{aligned} U(x,t) &= U_0 e^{-kx} \cos(\omega t - bx)\,, \\ U_x(x,t) &= U_0 e^{-kx} \left[b \sin(\omega t - bx) - k \cos(\omega t - bx) \right], \\ U_t(x,t) &= -\omega U_0 e^{-kx} \sin(\omega t - bx) \\ U_{xx}(x,t) &= U_0 e^{-kx} \Big[-2kb \sin(\omega t - bx) + (k^2 - b^2) \cos(\omega t - bx) \Big], \\ U_{tt}(x,t) &= -\omega^2 U_0 e^{-kx} \cos(\omega t - bx) \end{aligned}$$

Plugging it into the differential equation with $\alpha = \beta = c = 1$ gives

$$U_0 e^{-kx} \Big\{ \cos(\omega t - bx) \Big[-\omega^2 - (k^2 - b^2) + 1 \Big] + \sin(\omega t - bx) \left[2kb - 2\omega \right] \Big\} = 0\,! \qquad x, t > 0$$

So it follows

$kb = \omega$ and $-\omega^2 - k^2 + b^2 + 1 = 0$

$\implies k^2 b^2 + k^2 - b^2 - 1 = (k^2 - 1)(b^2 + 1) = 0 \quad$ where $b \in \mathbb{R}$

$\implies k^2 = 1, \quad$ with assumption that $k \in \mathbb{R}^+$, so $\quad k = 1\,.$

Hence we have that $kb = \omega$ and $\quad b = \omega$ and

$$U(x,t) = U_0 e^{-x} \cos(\omega(t - x))\,.$$