**Fachbereich Mathematik der Universität Hamburg** **SuSe 2022**
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# Differential Equations II for Engineering Students

## Work sheet 6

**Exercise:**

Given the following initial boundary value problem for $u = u(x,t)$:

$$\begin{aligned} &u_t - u_{xx} = e^{-t}\sin(2x) + 1 && x \in (0,\pi)\ , t \in \mathbb{R}^+, \\ &u(x,0) = \frac{1}{2}\sin(2x) && x \in (0,\pi)\,, \\ &u(0,t) = u(\pi,t) = t && t \in \mathbb{R}^+\,. \end{aligned}$$

a) Perform the homogenization of the boundary values.
What initial boundary value problem does one obtain after homogenizing the boundary values?

b) Solve for following the initial and boundary data:

(i)

$$\begin{aligned} &v^*_t - v^*_{xx} = 0 && x \in (0,\pi)\ , t \in \mathbb{R}^+, \\ &v^*(x,0) = \frac{1}{2}\sin(2x) && x \in (0,\pi)\,, \\ &v^*(0,t) = v^*(\pi,t) = 0 && t \in \mathbb{R}^+\,. \end{aligned}$$

and

$$\begin{aligned} &v^{**}_t - v^{**}_{xx} = e^{-t}\sin(2x) && x \in (0,\pi)\ , t \in \mathbb{R}^+, \\ &v^{**}(x,0) = 0 && x \in (0,\pi)\,, \\ &v^{**}(0,t) = v^{**}(\pi,t) = 0 && t \in \mathbb{R}^+\,. \end{aligned}$$

c) Provide the solution to the initial boundary value problem from part a).

**Solution:**

a) With $\quad v(x,t) = u(x,t) - t - \dfrac{x-0}{\pi-0}(t-t) = u(x,t) - t$

or $u(x,t) = v(x,t) + t$ we obtain

$u_t = v_t + 1, \quad u_{xx} = v_{xx}\,.$ New diffrerential equation:

$v_t + 1 - v_{xx} = e^{-t}\sin(2x) + 1 \iff \boxed{v_t - v_{xx} = e^{-t}\sin(2x).}$

Initial values: $v(x,0) = u(x,0) - 0 \overset{!}{=} \dfrac{1}{2}\sin(2x) \qquad x \in (0,\pi)\,.$

Boundary values : $v(0,t) = v(\pi,t) = t - t = 0\quad .$

b) (i) For the homogeneous differential equation with homogeneous boundary data, $\omega = 1\,,\ c = 1$ and given initial values

$$v^*(x,0) = \frac{1}{2}\sin(2x), \qquad x \in (0,\pi)\,,$$

one obtains
$$v^*(x,t) = \sum_{k=1}^{\infty} a_k e^{-c\omega^2 k^2 t}\sin(k\omega x) \;=\; \sum_{k=1}^{\infty} a_k\, e^{-k^2 t}\sin(kx)$$

From the initial data we have

$$v^*(x,0) = \sum_{k=1}^{\infty} a_k \sin(kx) \;=\; \frac{1}{2}\sin(2x).$$

The $a_k$ are the Fourier coefficients of $\frac{1}{2}\sin(2x)$.

$$a_k \;=\; \frac{2}{\pi}\int_0^{\pi}\frac{1}{2}\sin(2x)\sin(kx)dx. \tag{1}$$

Here you can omit the integration and do the coefficient comparison instead: $a_2 \;=\; \frac{1}{2}$ and $a_k = 0$ else. So

$$\boxed{v^*(x,t) \;=\; \frac{1}{2}e^{-4t}\sin(2x) \qquad .}$$

Of course, one gets the same result if one computes the Fourier coefficients via (1) by means of integration.

(ii) Inhomogeneous differential equation with homogeneous initial and boundary data

$$\begin{aligned}
&v^{**}_t - v^{**}_{xx} = e^{-t}\sin(2x) && x \in (0,\pi)\,, t \in \mathbb{R}^+,\\
&v^{**}(x,0) = 0 && x \in (0,\pi)\,,\\
&v^{**}(0,t) = v^{**}(\pi,t) = 0 && t \in \mathbb{R}^+\,.
\end{aligned}$$

Ansatz:

$$v^{**} \;=\; \sum_{k=1}^{\infty} a_k(t)\sin(kx)\,, \qquad a_k(0) = 0$$

Plugging it into the differential equation results in

$$\sum_{k=1}^{\infty}\left[\dot{a}_k(t) + k^2 a_k(t)\right]\sin(kx) \;=\; e^{-t}\sin(2x)$$

Hence we obtain $a_k(t) \equiv 0$ for $k \neq 2$ and the ordinary differential equation

$$\dot{a}_2(t) + 4a_2(t) \;=\; e^{-t}$$

for $a_2$. The solution to the associated homogeneous equation is

$$a_{2,h}(t) = Ce^{-4t}$$

The ansatz $a_2(t) = C(t)e^{-4t}$ gives

$$\dot{C}(t)\, e^{-4t} \;=\; e^{-t} \qquad \Longleftrightarrow \qquad C(t) = c + \frac{1}{3}\, e^{3t} \text{ z.B. } a_{2,p}(t) = \frac{1}{3}\, e^{-t}$$

$$a_2(t) = c\, e^{-4t} + \frac{1}{3}\, e^{-t} \qquad \text{and with } a_2(0) = 0 \text{ it follows } c = -1/3$$

$$a_2(t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)$$

$$v^{**}(x,t) = \frac{1}{3}\left(e^{-t} - e^{-4t}\right)\sin(2x)$$

c) With the notation from a) and b) it holds

$$v(x,t) = v^*(x,t) + v^{**}(x,t) = \frac{1}{6}\left(2e^{-t} + e^{-4t}\right)\sin(2x)\,\cdot$$

and

$$u(x,t) \;=\; v(x,t) + t \;=\; \frac{1}{6}\left(2e^{-t} + e^{-4t}\right)\sin(2x) \;+\; t\,.$$

**Discussion: 26.06 - 30.06.2023**