Fachbereich Mathematik der Universität Hamburg **SuSe 2022**
Prof. Dr. J. Behrens
Dr. H. P. Kiani
S. Onyshkevych

Differential Equations II for Engineering Students

Homework sheet 6

Exercise 1: (Exam, Prof. Behrens 2022, 7 Points)

a) Given the initial boundary value problem

$$\begin{aligned} u_t - 5u_{xx} &= \frac{\pi x}{4}\sin(\pi t) && \text{for } x \in (0,4), t > 0,\\ u(x,0) &= 2\sin(\pi x) + 3\sin(2\pi x) && \text{for } x \in [0,4],\\ u(0,t) = 0, \quad u(4,t) &= 1 - \cos(\pi t) && \text{for } t > 0. \end{aligned}$$

Transform the problem into an initial boundary value problem with homogeneous boundary data using a suitable homogenization of the boundary conditions.

b) Solve the following initial boundary value problem:

$$\begin{aligned} v_t - 5v_{xx} &= 0 && \text{for } x \in (0,4), t > 0,\\ v(x,0) &= 2\sin(\pi x) + 3\sin(2\pi x) && \text{for } x \in [0,4],\\ v(0,t) = 0, \quad v(4,t) &= 0 && \text{for } t > 0. \end{aligned}$$

c) Provide the solution to the initial boundary value problem from part a).

Solution:

a) Homogenization:

$$v(x,t) = u(x,t) - 0 - \frac{x}{4}(1 - \cos(\pi t) - 0) = u(x,t) - \frac{x}{4}(1 - \cos(\pi t))$$

or

$$u(x,t) = v(x,t) + \frac{x}{4}(1 - \cos(\pi t)). \qquad \textbf{(1 point)}$$

Then it holds:

$$u_t = v_t + \frac{\pi x}{4}\sin(\pi t), \qquad v_{xx} = u_{xx}$$

New differential equation: $v_t + \frac{\pi x}{4}\sin(\pi t) - 5v_{xx} = \frac{\pi x}{4}\sin(\pi t) \iff$

$$\boxed{v_t - 5v_{xx} = 0} \qquad \textbf{(1 point)}$$

Initial data:

$$\begin{aligned} v(x,0) &= u(x,0) - \frac{x}{4}(1 - \cos(0))\\ &= 2\sin(\pi x) + 3\sin(2\pi x) - 0 \iff \end{aligned}$$

$$\boxed{v(x,0) = 2\sin(\pi x) + 3\sin(2\pi x)}$$

Boundary data: $\boxed{v(0,t) = v(4,t) = 0}$ **(1 point)**

b) With $\omega = \frac{\pi}{4}$ and $c = 5$ it holds:

$$v(x,t) \;=\; \sum_{k=1}^{\infty} a_k e^{-c\omega^2 k^2 t} \sin(k\omega x) \;=\; \sum_{k=1}^{\infty} a_k e^{-\frac{5k^2\pi^2}{16}t} \sin(\frac{k\pi}{4}x) \qquad \textbf{(1 point)}$$

Inserting the initial values gives:

$v(x,0) \;=\; \sum_{k=1}^{\infty} a_k \sin(\frac{k\pi}{4}x) \;\overset{!}{=}\; 2\sin(\pi x) \;+\; 3\sin(2\pi x)$

$\Longrightarrow a_4 = 2,\; a_8 = 3,\; a_k = 0 \qquad \forall k \notin \{4, 8\}.$

$$v(x,t) \;=\; 2\,e^{-\frac{5\cdot 4^2\pi^2}{16}t} \sin(\frac{4\pi}{4}x) \;+\; 3\,e^{-\frac{5\cdot 8^2\pi^2}{16}t} \sin(\frac{8\pi}{4}x) \qquad \textbf{(2 points)}$$

c) For the solution of a) we thus get

$$u(x,t) = \; v(x,t) + \frac{x}{4}\,(1 - \cos(\pi t))$$

$$= \; 2\,e^{-5\pi^2 t} \sin(\pi x) \;+\; 3\,e^{-20\pi^2 t} \sin(2\pi x) + \frac{x}{4}(1 - \;\cos(\pi t)). \qquad \textbf{(1 point)}$$

Exercise 2:

a) Using a product ansatz, derive the series representation given in lecture 10 (page 18) for the solution of the following Neumann problem.

$$\begin{aligned} u_t &= u_{xx}, & 0 < x < 1,\ t > 0, \\ u(x,0) &= g(x), & 0 < x < 1, \\ u_x(0,t) &= u_x(1,t) = 0 & t > 0. \end{aligned}$$

b) Solve the initial boundary value problem a) with $g(x) = 2\pi x - \sin(2\pi x)$.

Hint: $2\sin(\alpha)\cdot\cos(\beta) = \sin(\alpha+\beta) + \sin(\alpha-\beta)$.

Solution:

a) Short version: From the lecture we know that the ansatz $u_k(x,t) = v_k(x)\cdot w_k(t)$ with $L = 1$ leads to

$$v_k(x) = \cos(k\pi x), \qquad \text{and} \qquad w_k(t) = e^{-k^2\pi^2 t}, \qquad k \in \mathbb{N}_0$$

Very long version: The ansatz $u(x,t) = v(x)\cdot w(t)$ *yields:*

$v'' = -\lambda\, v, \quad \dot{w} = -\lambda w, \quad v'(0) = v'(1) = 0\,.$

Case distinction under the condition that the solution does not vanish:

$$\begin{aligned} \lambda = 0 &\Longrightarrow v(x) = a_0 + b_0 x, \quad v' = b_0 = 0 \\ &\Longrightarrow v_0(x) = a_0\,. \\ \lambda < 0 &\Longrightarrow v(x) = a e^{\sqrt{-\lambda}x} + b e^{-\sqrt{-\lambda}x} \\ &\qquad v'(0) = 0 \Longleftrightarrow a = b \\ &\qquad v'(1) = 0 \Longleftrightarrow a\sqrt{-\lambda}(e^{\sqrt{-\lambda}} - e^{-\sqrt{-\lambda}}) = 0 \\ &\qquad \Longleftrightarrow (u \equiv 0) \vee (e^{\sqrt{\lambda}} = e^{-\sqrt{\lambda}} \Longleftrightarrow \lambda = 0) \qquad \textit{Contradiction!} \\ \lambda > 0 &\Longrightarrow v(x) = a\cos(\sqrt{\lambda}\,x) + b\sin(\sqrt{\lambda}\,x) \\ &\qquad v'(x) = (\sqrt{\lambda})(-a\sin(\sqrt{\lambda}\,x) + b\cos(\sqrt{\lambda}\,x) \\ &\qquad v'(0) = 0 \Longleftrightarrow b = 0 \\ &\qquad v'(1) = 0 \Longleftrightarrow (u \equiv 0) \vee (\sin(\sqrt{\lambda}) = 0 \Longleftrightarrow \lambda_k = k^2\pi^2). \end{aligned}$$

So overall we get

$$v_k(x) = \cos(k\pi x), \quad k \in \mathbb{N}_0\,.$$

One can easily calculate for the time component

$$w_k(t) = e^{-k^2\pi^2 t}, \quad k \in \mathbb{N}_0\,.$$

So as a series representation for the solution one has

$$u(x,t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k e^{-k^2\pi^2 t}\cos(k\pi x)\,.$$

To fulfill:

$$u(x,0) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos(k\pi x)\,.$$

To determine the coefficients, g is continued evenly and $2-$ periodically and the Fourier coefficients are determined

$$a_k = 2\int_0^1 g(x)\cos(k\pi x)\,dx\,.$$

b) For $k \notin \{0,2\}$ one computes for $g(x) = 2\pi x - \sin(2\pi x)$.

$$\begin{aligned}
a_k &= 2\int_0^1 (2\pi x - \sin(2\pi x))\cos(k\pi x)\,dx \\
&= 2\int_0^1 2\pi x\cos(k\pi x)\,dx - 2\int_0^1 \sin(2\pi x)\cos(k\pi x)\,dx \\
&= 4\pi x\,\frac{\sin(k\pi x)}{k\pi}\bigg|_0^1 - 4\pi\int_0^1 \frac{\sin(k\pi x)}{k\pi}\,dx - \int_0^1 \sin(2\pi x + k\pi x) + \sin(2\pi x - k\pi x)\,dx \\
&= \frac{4}{k^2\pi}\cos(k\pi x)|_0^1 + \frac{\cos((k+2)\pi x)}{(k+2)\pi}\bigg|_0^1 + \frac{\cos((-k+2)\pi x)}{(-k+2)\pi}\bigg|_0^1 \\
&= \frac{4}{k^2\pi}(\cos(k\pi) - 1) + \frac{1}{(k+2)\pi}(\cos((k+2)\pi) - 1) - \left(\frac{1}{(k-2)\pi}\cos((-k+2)\pi) - 1\right) \\
&= \frac{4}{k^2\pi}\left((-1)^k - 1\right) - \left(\frac{1}{(k-2)\pi} - \frac{1}{(k+2)\pi}\right)(\cos(k\pi) - 1) \\
&= \left(\frac{4}{k^2\pi} - \frac{4}{(k^2-4)\pi}\right)\cdot\left((-1)^k - 1\right) = \left(\frac{16\cdot\left(1-(-1)^k\right)}{k^2\cdot(k^2-4)\pi}\right)
\end{aligned}$$

For $k = 0$ we have

$$a_0 = 2\int_0^1 2\pi x - \sin(2\pi x)\,dx = 2\pi$$

and for $k = 2$

$$\begin{aligned}
a_2 &= 2\int_0^1 2\pi x\cos(2\pi x) - \sin(2\pi x)\cos(2\pi x)dx \\
&= \frac{4}{2^2\pi}\left((-1)^2 - 1\right) - \int_0^1 \sin(4\pi x)dx = 0.
\end{aligned}$$