Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Work sheet 5

Exercise 1:
Determine the solution to the following problem

$$\Delta v = 0 \quad \text{for } 0 \le x^2 + y^2 < 9,$$
$$v(x,y) = \frac{x}{9}\,(x-y) \quad \text{on } x^2 + y^2 = 9\,.$$

Hint:

- *Use polar coordinates and an appropriate product approach.*
- $\cos(2\phi) = 2\cos^2(\phi) - 1$, $\sin(2\phi) = 2\sin(\phi)\cos(\phi)$.

Solution:

We use polar coordinates $x = r\cos(\phi)$, $y = r\sin(\phi)$, and
$v(x(r,\phi), y(r,\phi)) = u(r,\phi)$.
Then the boundary condition is:

$$u(3,\phi) = v(x(3,\phi), y(3,\phi)) = v(3\cos(\phi), 3\sin(\phi)) = \frac{3\cos(\phi)}{9}\,(3\cos(\phi) - 3\sin(\phi)) \ .$$

We know the general solution from the auditorium exercise class:

$$\boxed{u(r,\phi) = c_0 + d_0 \ln(r) + \sum_k (c_k r^{-k} + d_k r^k)(a_k \cos(k\phi) + b_k \sin(k\phi))}$$

Since the solution has to be defined inside a circle centered at zero (particularly also bounded), the negative powers and the ln − term are out of the question. Therefore we get w.l.o.g. with $d_k = 1$ the representation of the solution:

$$u(r,\phi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\phi) + b_k \sin(k\phi))\, r^k\,.$$

We also have the condition from boundary data

$$u(3,\phi) = a_0 + \sum_{k=1}^{\infty} (a_k \cos(k\phi) + b_k \sin(k\phi))\, 3^k$$
$$\overset{!}{=} \cos(\phi)\,(\cos(\phi) - \sin(\phi)) = \cos^2(\phi) - \sin(\phi)\cos(\phi) = \frac{1}{2}\,(1 + \cos(2\phi) - \sin(2\phi)) \ .$$

A comparison of coefficients then gives

$a_0 + \left(a_2 \cos(2\phi) + b_2 \sin(2\phi)\right) \cdot 3^2 \overset{!}{=} \frac{1}{2}\left(1 + \cos(2\phi) - \sin(2\phi)\right)$

$\Longrightarrow a_0 = \frac{1}{2}$ and $a_2 = -b_2 = \dfrac{1}{18}$ and hence the solution is

$$u(r, \phi) = \frac{1}{2} + \frac{r^2}{18}\cos(2\phi) - \frac{r^2}{18}\sin(2\phi)\,.$$

Exercise 2:

Solve the following Dirichlet problems

a)

$$\begin{aligned} \Delta v &= 0 && \text{in } (0,2)\times(0,1),\\ v(x,0) &= \sin(\pi x), && x\in(0,2),\\ v(x,1) &= 0, && x\in(0,2),\\ v(0,y) &= 0, && y\in(0,1)\\ v(2,y) &= 0, && y\in(0,1). \end{aligned}$$

b)

$$\begin{aligned} \Delta w &= 0 && \text{in } (0,2)\times(0,1),\\ w(x,0) &= 0, && x\in(0,2),\\ w(x,1) &= -5\sin(2\pi x), && x\in(0,2),\\ w(0,y) &= 0, && y\in(0,1)\\ w(2,y) &= 0, && y\in(0,1). \end{aligned}$$

c)

$$\begin{aligned} \Delta u &= 0 && \text{in } (0,2)\times(0,1),\\ u(x,0) &= 6\sin(\pi x), && x\in(0,2),\\ u(x,1) &= 5\sin(2\pi x), && x\in(0,2),\\ u(0,y) &= 0, && y\in(0,1)\\ u(2,y) &= 0, && y\in(0,1). \end{aligned}$$

Solution:

a) A product ansatz $v(x,y) = X(x)Y(y)$ leads to the ordinary differential equations

$$X'' = -\lambda X, \quad Y'' = \lambda Y$$

For non-trivial solutions $v \not\equiv 0$ we have from the boundary data

$$\begin{aligned} v(0,y) &= X(0)Y(y) = 0 \Longrightarrow X(0) = 0,\\ v(2,y) &= X(2)Y(y) = 0 \Longrightarrow X(2) = 0,\\ v(x,1) &= X(x)Y(1) = 0 \Longrightarrow Y(1) = 0. \end{aligned}$$

Because of the two zero boundary values for X we first solve the boundary value problem $X'' = -\lambda X$, $X(0) = 0$, $X(2) = 0$.

This way we get non-trivial solutions (as on work sheet 1) only for positive λ

$$\begin{aligned} &X(x) = A_\lambda \cos(\sqrt{\lambda} x) + B_\lambda \sin(\sqrt{\lambda} x)\\ &X(0) = 0 \Longrightarrow A_\lambda = 0, \quad X(2) = 0 \Longrightarrow 2\sqrt{\lambda_k} = k\pi\\ &X_k(x) = \sin(\frac{k\pi}{2} x), \ k \in \mathbb{N} \end{aligned}$$

With these values of λ we are solving the differential equation for Y.

$$Y'' = \left(\frac{k\pi}{2}\right)^2 Y \Longrightarrow Y_k(y) = A_k e^{-\frac{k\pi}{2}y} + B_k e^{\frac{k\pi}{2}y}$$
$$Y_k(1) = 0 \Longrightarrow A_k e^{-\frac{k\pi}{2}} + B_k e^{\frac{k\pi}{2}} = 0 \Longrightarrow A_k = -e^{k\pi} B_k$$
$$\Longrightarrow Y_k(y) = B_k \left(e^{\frac{k\pi}{2}y} - e^{k\pi} e^{-\frac{k\pi}{2}y}\right)$$

With the help of the superposition principle we get the function $v(x,y)$ as a linear combination of the $X_k(x)Y_k(x)$, $k \in \mathbb{N}$ and without discussing the convergence make the series ansatz

$$v(x,y) = \sum_{k=1}^{\infty} c_k \left(e^{\frac{k\pi}{2}y} - e^{k\pi} e^{-\frac{k\pi}{2}y}\right) \sin(\frac{k\pi}{2}x).$$

We now get the coefficients c_k from the boundary condition that has not yet been used $v(x,0) = \sin(\pi x)$. It holds:

$$v(x,0) = \sum_{k=1}^{\infty} c_k \left(1 - e^{k\pi}\right) \sin(\frac{k\pi}{2}x) \stackrel{!}{=} \sin(\pi x)$$
$$\Longrightarrow c_2 \left(1 - e^{2\pi}\right) = 1,\ c_k = 0, k \neq 2$$

Altogether we have:

$$v(x,y) = \frac{e^{\pi y} - e^{2\pi} e^{-\pi y}}{1 - e^{2\pi}} \sin(\pi x).$$

b) For the second problem we get completely analogous

$X_k(x) = \sin(\frac{k\pi}{2}x)$, $\qquad Y_k(y) = A_k e^{-\frac{k\pi}{2}y} + B_k e^{\frac{k\pi}{2}} y$

From the third zero constraint we have

$Y_k(0) = 0 \Longrightarrow Y_k(y) = A_k + B_k = 0 \Longrightarrow B_k = -A_k$.

$$w(x,y) = \sum_{k=1}^{\infty} A_k \left(e^{-\frac{k\pi}{2}y} - e^{\frac{k\pi}{2}} y\right) \sin(\frac{k\pi}{2}x).$$

We now get the coefficients A_k from the boundary condition $w(x,1) = 5\sin(2\pi x)$, which has not yet been used. It holds:

$$w(x,1) = \sum_{k=1}^{\infty} A_k \left(e^{-\frac{k\pi}{2}} - e^{\frac{k\pi}{2}}\right) \sin(\frac{k\pi}{2}x) \stackrel{!}{=} -5\sin(2\pi x).$$
$$\Longrightarrow A_4 \left(e^{-2\pi} - e^{2\pi}\right) = -5,\ A_k = 0, k \neq 4$$

So we have:

$$w(x,y) = A_4 \cdot (e^{-\frac{4\pi}{2}y} - e^{\frac{4\pi}{2}y}) \sin(\frac{4\pi}{2}x) = 5 \cdot \frac{e^{-2\pi y} - e^{2\pi y}}{e^{2\pi} - e^{-2\pi}} \sin(2\pi x).$$

c) Because of the linearity of differential equation one gets $u = 6v - w$.

Discussion: 12-16.06.2023