Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
Prof. Dr. J. Behrens
Dr. H. P. Kiani

Differential Equations II for Engineering Students

Homework sheet 5

Exercise 1:

a) We are looking for a solution of the Laplace equation $\Delta v(x,y) = 0$ in a rotationally symmetric area, for example in a circle. The area can then be better described using polar coordinates. This is done as follows

$x = r\cos\phi\,,\ y = r\sin\phi\,$, and

$v(x(r,\phi), y(r,\phi)) \;=\; u(r,\phi)\;.$

Show that for $r \neq 0$ the following equivalence holds:

$$r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0 \iff r^2\left(v_{xx} + v_{yy}\right) \;=\; 0\,.$$

b) Find a solution to the following boundary value problem:

$$\begin{aligned} \Delta(v) &= 0 \quad \text{für } 1 < x^2+y^2 < 4,\\ v(x,y) &= 1 \quad \text{auf } x^2+y^2 = 1,\\ v(x,y) &= 2 \quad \text{auf } x^2+y^2 = 4. \end{aligned}$$

Hint: Use polar coordinates. The boundary data are independent of ϕ. So try the ansatz

$$v(x,y) = u(r,\phi) = w(r).$$

Solution:

a)

$$\begin{aligned} u_r &= v_x\cdot x_r + v_y\cdot y_r = \cos(\phi)v_x + \sin(\phi)v_y\\ u_\phi &= v_x\cdot x_\phi + v_y\cdot y_\phi = -r\sin(\phi)v_x + r\cos(\phi)v_y\\ u_{rr} &= v_{xx}\cos^2(\phi) + 2v_{xy}\cos(\phi)\sin(\phi) + v_{yy}\sin^2(\phi)\\ u_{\phi\phi} &= v_{xx}r^2\sin^2(\phi) + 2v_{xy}r^2\cos(\phi)(-\sin(\phi)) + v_{yy}r^2\cos^2(\phi) - r\cos(\phi)v_x - r\sin(\phi)v_y \end{aligned}$$

Plug into the differential equation

$$\begin{aligned} r^2u_{rr} + ru_r + u_{\phi\phi} &= (r^2\cos^2(\phi) + r^2\sin^2(\phi))v_{xx}\\ &\quad + (2r^2\cos(\phi)\sin(\phi) - 2r^2\cos(\phi)\sin(\phi)))v_{xy}\\ &\quad + (r^2\sin^2(\phi) + r^2\cos^2(\phi))v_{yy}\\ &\quad + r\cos(\phi)v_x + r\sin(\phi)v_y - r\cos(\phi)v_x - r\sin(\phi)v_y\\ &= r^2(v_{xx} + v_{yy})\,. \end{aligned}$$

b) We not just go over to polar coordinates, but because of the nature of the boundary conditions we make the ansatz $v(x,y) = u(r,\phi) = w(r)$. From part a) we obtain the differential equation

$$u_{rr} + \frac{1}{r}u_r + \frac{1}{r^2}u_{\phi\phi} = w'' + \frac{1}{r}w' = 0.$$

So we have an ordinary first order differntial equation for $g := w'$.

$g'(r) = -\frac{1}{r}g(r) \implies \frac{dg}{g} = -\frac{dr}{r} \implies \ln(|g|]) = -\ln(|r|]) + k$

$\implies g(r) = \dfrac{c}{r} = w'(r)$. From this we get

$$u(r,\phi) = w(r) = c\ln(r) + d.$$

From boundary data we get

$$\begin{aligned} u(1,\phi) =& 1 \implies c\ln(1) + d = 1 \implies & d = 1. \\ u(2,\phi) =& 2 \implies c\ln(2) + 1 = 2 \implies & c = \frac{1}{\ln(2)}. \\ u(r,\phi) =& \frac{1}{\ln(2)}\ln(r) + 1. \\ v(x,y) =& \frac{\ln\sqrt{x^2+y^2}}{\ln(2)} + 1. \end{aligned}$$

Exercise 2:

a) Show that through $a_k = 0, \forall k \in \mathbb{N}_0, \quad \beta_k = \begin{cases} 0 & \text{for } k \in \mathbb{N} \text{ even,} \\ -\dfrac{8}{(k\pi)^3} & \text{for } k \in \mathbb{N} \text{ odd} \end{cases}$

the Fourier coefficients of the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\pi y) + \beta_k \sin(k\pi y)\right)$$

of the odd, 2–periodic continuation of

$$g(y) = y^2 - y, \; 0 \le y \le 1$$

are given.

b) Determine with the help of an appropriate product ansatz and by using a) the solution of the following boundary value problem

$$\begin{array}{ll} \Delta u(x,y) = 0 & x \in (0,1), \; y \in (0,1), \\ u(x,0) = 0 & x \in [0,1]\,, \\ u(x,1) = 0 & x \in [0,1]\,, \\ u(0,y) = g(y) = y^2 - y & y \in [0,1]\,, \\ u(1,y) = 0 & y \in [0,1]\,. \end{array}$$

Solution:

a) We have that $a_k = 0$, since the continued function is odd. For β_k one gets

$$\begin{aligned} \beta_k &= 2\int_0^1 (y^2 - y)\sin(k\pi y)dy \;=\; 2\left[(y^2-y)\frac{-\cos(k\pi y)}{k\pi}\right]_0^1 + 2\int_0^1 (2y-1)\,\frac{\cos(k\pi y)}{k\pi}dy \\ &= \frac{2}{k\pi}\left[(2y-1)\frac{\sin(k\pi y)}{k\pi}\right]_0^1 - \frac{4}{k\pi}\int_0^1 \frac{\sin(k\pi y)}{k\pi}dy \\ &= \frac{4}{(k\pi)^2}\left[\frac{\cos(k\pi y)}{k\pi}\right]_0^1 = \begin{cases} 0 & \text{for } k \text{ even} \\ -\dfrac{8}{(k\pi)^3} & \text{for } k \text{ odd} \end{cases} \end{aligned}$$

b) Inserting the product ansatz $u(x,y) = v(x)w(y)$ in the differential equation gives

$$v''(x)w(y) + v(x)w''(y) = 0 \implies \frac{v''}{v} = -\frac{w''}{w} = \lambda \qquad \lambda \text{ constant}\,.$$

The boundary values $u(x,0)$ and $u(x,1) = 0$ give $w(0) = w(1) = 0$. The solutions to the eigenvalue problem

$$w'' = -\lambda w, \quad w(0) = w(1) = 0$$

are according to sheet 1, classroom exercise 1

$$w_k(y) \; = \; c_k \sin(k\pi y), \qquad \text{where } \lambda_k = k^2\pi^2.$$

The second differential equation $\dfrac{v''}{v} = k^2\pi^2$ has the solutions

$$v_k(x) = \tilde{a}_k e^{k\pi x} + \tilde{b}_k e^{-k\pi x}.$$

Every function $u_k(x,y) \; = \; v_k(x) \cdot w_k(y)$ solves the linear differential equation and thus also every finite linear combination of these solutions. Without discussing the convergence, we make the ansatz

$$u(x,y) \; = \; \sum_{k=1}^{\infty} \sin(k\pi y) \left(\tilde{a}_k e^{k\pi x} + \tilde{b}_k e^{-k\pi x} \right).$$

From the not yet used boundary condition $u(1,y) \; = \; v(1)w(y) = 0$ it follows
$\tilde{a}_k e^{k\pi} + \tilde{b}_k e^{-k\pi} = 0 \iff \tilde{b}_k = -\tilde{a}_k e^{2k\pi}$
and hence
$$u(x,y) \; = \; \sum_{k=1}^{\infty} \tilde{a}_k \sin(k\pi y) \left(e^{k\pi x} - e^{2k\pi}\, e^{-k\pi x} \right).$$

The last boundary condition is:
$$u(0,y) \; = \; \sum_{k=1}^{\infty} \tilde{a}_k \sin(k\pi y) \left(1 - e^{2k\pi} \right) \; = \; y^2 - y \; .$$
With the Fourier coefficients β_k of the function $y^2 - y$ when expanding according to the functions $\sin(k\pi y)$ from part a) it holds
$$\beta_k \; = \; \tilde{a}_k \left(1 - e^{2k\pi} \right)$$
or
$$\tilde{a}_k \; = \; \frac{\beta_k}{1 - e^{2k\pi}} \, .$$

and hence

$$\begin{aligned} u(x,y) \; &= \; \sum_{k=1}^{\infty} \frac{\beta_k}{1 - e^{2k\pi}} \sin(k\pi y) \left(e^{k\pi x} - e^{2k\pi}\, e^{-k\pi x} \right) \\ &= \; \sum_{k=1}^{\infty} \frac{\beta_k}{e^{-k\pi} - e^{k\pi}} \sin(k\pi y) \left(e^{k\pi x - k\pi} - e^{-k\pi x + k\pi} \right). \\ &= \; \sum_{k=1}^{\infty} \frac{\beta_k}{e^{-k\pi} - e^{k\pi}} \sin(k\pi y) \left(e^{k\pi(x-1)} - e^{-k\pi(x-1)} \right). \end{aligned}$$

Those who prefer to work with the hyperbolic functions get
$$u(x,y) \; = \; \sum_{k=1}^{\infty} \frac{\beta_k}{-\sinh(k\pi)} \sin(k\pi y) \cdot \sinh(k\pi(x-1)).$$