Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Work sheet 4

Exercise 1:

Given the partial differential equation

$$3u_{xx} + 8u_{xt} - 3u_{tt} = 0 \quad \text{for } x \in \mathbb{R}, t > 0$$

a) Determine the type of the differential equation (hyperbolic, parabolic or elliptic).

b) Transform the differential equation into the diagonal form $\alpha \cdot \tilde{u}_{\eta\eta} + \beta \cdot \tilde{u}_{\tau\tau} = 0$.

c) How do the new coordinates η, τ depend on the old coordinates t, x?

Solution:

$$3u_{xx} + 8u_{xt} - 3u_{tt} = 0 \quad \text{for } x \in \mathbb{R}, t > 0\,.$$

a) From

$$A = \begin{pmatrix} 3 & 4 \\ 4 & -3 \end{pmatrix} \Longrightarrow \det(A) = -9 - 16 < 0$$

it follows that it is a hyperbolic differential equation.

b) Eigenvalues of A:

$(3 - \lambda)(-3 - \lambda) - 16 = 0 \Longrightarrow \lambda^2 - 25 = 0 \Longrightarrow \lambda_{1,2} = \mp 5\,.$

Thus, as a diagonal form, one obtains

$$-5\tilde{u}_{\eta\eta} + 5\tilde{u}_{\tau\tau} = 0$$

c) Normalized eigenvectors of A:

$$(\boldsymbol{A} - \lambda_1 I)\,\boldsymbol{v} = 0 \Longleftrightarrow \begin{pmatrix} 8 & 4 \\ 4 & 2 \end{pmatrix} \boldsymbol{v} = 0 \Longleftarrow \boldsymbol{v} = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 \\ -2 \end{pmatrix}$$

Orthogonality property and an analogous calculation for λ_2 leads to

$$(A - \lambda_2 I)w = 0 \Longrightarrow w = \frac{1}{\sqrt{5}} \begin{pmatrix} 2 \\ 1 \end{pmatrix}$$

Hence one obtains the transformation matrix and the new coordinates

$$S = \frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 2 \\ -2 & 1 \end{pmatrix}, \quad \begin{pmatrix} \tau \\ \eta \end{pmatrix} = S^T \begin{pmatrix} t \\ x \end{pmatrix}$$

so

$$\tau = \frac{1}{\sqrt{5}}(x - 2t)\,, \qquad \eta = \frac{1}{\sqrt{5}}(2x + t)\,.$$

Exercise 2:

Determine the value of the harmonic in $\Omega := \{\binom{x}{y} \in \mathbb{R}^2 : x^2 + y^2 < 4\}$ function $u(x, y)$ in the point $\binom{0}{0}$ with the boundary data

a) $u(x, y) = \dfrac{x + y + 1}{4}$ on the boundary of $\Omega = \partial\Omega$ using the Poisson integral representation of the solution.

b) $u(x, y) = x^2 y + 2$ on $\partial\Omega$, using the mean value property of harmonic functions.

c) $u(x, y) = x^2 - y^2$ on $\partial\Omega$, using the uniqueness property of the solution.

d) $u(x, y) = x^2 + y^2$ on $\partial\Omega$, without calculation, using the maximum/minimum principle.

Solution:

a) $u(x, y) = \dfrac{x + y + 1}{4}$ on the boundary of $\Omega = \partial\Omega$ using the Poisson integral representation of the solution.

K_2 is the circle of the radius $R = 2$ and $c(t) = (2\cos\phi,\, 2\sin\phi)$ is a parametrization of K_2. Then, from Poisson's integral formula it follows

$$u(x, y) = \frac{R^2 - x^2 - y^2}{2\pi R} \int_{\|z\|=R} \frac{g(\boldsymbol{z})}{\|\,\boldsymbol{z} - \boldsymbol{x}\,\|^2}\, d\boldsymbol{z}$$

$$u(0, 0) = \frac{4}{4\pi} \int_{\|z\|=2} \frac{z_1 + z_2 + 1}{16}\, d(z_1, z_2) = \frac{1}{4\pi} \int_0^{2\pi} \frac{2\cos\phi + 2\sin\phi + 1}{4} \cdot 2\, dt = \frac{1}{4}.$$

b) $u(x, y) = x^2 y + 2$ on $\partial\Omega$, using the mean value property of harmonic functions.

Let K_2 and $c(t)$ be defined as in part a). Then from the mean value property it follows

$$\begin{aligned} u(0, 0) &= \frac{1}{2\pi \cdot 2} \int_{K_2} (x^2 y + 2)\, d(x, y) = \frac{1}{4\pi} \int_0^{2\pi} (4\cos^2(\phi) 2\sin(\phi) + 2) \cdot 2\, dt \\ &= \frac{1}{2\pi} \left[-\frac{8\cos^3(\phi)}{3} + 2t \right]_0^{2\pi} = 2. \end{aligned}$$

c) $u(x, y) = x^2 - y^2$ on $\partial\Omega$, using the uniqueness property of the solution.

$u(x, y) = x^2 - y^2$ solves the potential equation in the whole disk, so it is the unique solution. It follows that $u(0, 0) = 0$

d) $u(x, y) = x^2 + y^2$ on $\partial\Omega$, without calculation, using the maximum/minimum principle.

$u(x.y)$ is constant on the boundary of Ω. Since the maximum and minimum are assumed to be on the boundary, u is constant $= 4$ throughout the disc.