Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Homework sheet 4

Exercise 1: Determine the type of the following partial differential equations

a) $2u_{xx} - 8u_{xy} + 8u_{yy} + u_y = u\,,$

b) $2u_{xy} + u_{yy} + xu_x = \cos(y)\,,$

c) $3u_{xx} + 2u_{xy} + u_{yy} = 0\,,$

d) $u_{xx} + e^x u_{yy} + \sin(x)(u_x + u_y) = y + x\,,$

e) $(x^2 + y^2)u_{xx} + 2(x + y)u_{xy} + u_{yy} = 0\,.$

Solution :

a) $2u_{xx} - 8u_{xy} + 8u_{yy} + u_y = u$

$$2 \cdot 8 - 4^2 = 0 \qquad \text{parabolic}\,.$$

b) $2u_{xy} + u_{yy} + xu_x = \cos(y)$

$$1 \cdot 0 - 1 = -1 \qquad \text{hyperbolic}\,.$$

c) $3u_{xx} + 2u_{xy} + u_{yy} = 0$

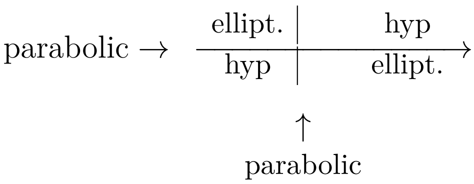
$$3 \cdot 1 - 1^2 = 2 \qquad \text{elliptic}\,.$$

d) $u_{xx} + e^x u_{yy} + \ldots = \ldots$

$$1 \cdot e^x - 0^2 > 0 \qquad \text{elliptic}\,.$$

e) $(x^2 + y^2)u_{xx} + 2(x + y)u_{xy} + u_{yy} = 0$

$$x^2 + y^2 - (x+y)^2 = -2xy \qquad \begin{cases} \text{parabolic for} & xy = 0, \\ \text{hyperbolic for} & xy > 0, \\ \text{elliptic for} & xy < 0. \end{cases}$$



Exercise 2: Given the initial value problem

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} &= 0 \quad \text{for } x \in \mathbb{R},\, t \in \mathbb{R}^+ \\ u(x,0) &= 0 \quad \text{for } x \in \mathbb{R}, \\ u_t(x,0) &= 2xe^{-x^2}. \quad \text{for } x \in \mathbb{R}. \end{aligned}$$

Solve the problem using substitution $\alpha = x + \frac{t}{4},\ \mu = x - t$.
Hint: Compute $v_{\alpha\mu}$ *for* $v(\alpha, \mu) := u(x(\alpha,\mu), t(\alpha,\mu))$.
Alternatively: convert the derivatives in terms of x, t *into derivatives in terms of* α, μ.
Solution:

Using the substitution $\alpha = x + \frac{t}{4}$, $\mu = x - t$ we obtain

$$x = \frac{4\alpha + \mu}{5}, \quad t = \frac{4\alpha - 4\mu}{5},$$

so

$$u(x,t) = u(x(\alpha,\mu), t(\alpha,\mu)) = u\left(\frac{4\alpha+\mu}{5}, \frac{4\alpha - 4\mu}{5}\right) =: v(\alpha,\mu)$$

First approach for the transformation of the differential equation

$$\begin{aligned} v_\alpha &= u_x \cdot \frac{dx}{d\alpha} + u_t \cdot \frac{dt}{d\alpha} = \frac{4}{5}u_x + \frac{4}{5}u_t \\ v_{\alpha\mu} &= \left(u_{xx} \cdot \frac{dx}{d\alpha} \cdot \frac{dx}{d\mu} + u_{xt} \cdot \frac{dx}{d\alpha} \cdot \frac{dt}{d\mu}\right) + \left(u_{tx} \cdot \frac{dt}{d\alpha} \cdot \frac{dx}{d\mu} + u_{tt} \cdot \frac{dt}{d\alpha} \cdot \frac{dt}{d\mu}\right) \\ &= \frac{4}{25}u_{xx} - \frac{16}{25}u_{xt} + \frac{4}{25}u_{tx} - \frac{16}{25}u_{tt} = \frac{4}{25}\left(u_{xx} - 4u_{xt} + u_{xt} - 4u_{tt}\right) \end{aligned}$$

For every twice continuously differentiable solution of the original differential equation using the introduced notation we have $\boxed{v_{\alpha\mu} = 0.}$

Alternative way of transformation: For twice continuously differentiable functions u and v with the notation introduced above we have $\alpha = x + \frac{t}{4},\ \mu = x - t$ and $u(x,t) =: v(\alpha,\mu) = v(\alpha(x,t), \mu(x,t))$:

$$\begin{aligned} u_x &= v_\alpha \cdot \alpha_x + v_\mu \cdot \mu_x = v_\alpha + v_\mu \\ u_t &= v_\alpha \cdot \alpha_t + v_\mu \cdot \mu_t = \frac{1}{4}v_\alpha - v_\mu \\ u_{xx} &= v_{\alpha\alpha}\alpha_x + v_{\alpha\mu}\mu_x \cdot \ + v_{\mu\alpha} \cdot \alpha_x + v_{\mu\mu} \cdot \mu_x = v_{\alpha\alpha} + 2v_{\alpha\mu} + v_{\mu\mu} \\ u_{xt} &= v_{\alpha\alpha} \cdot \alpha_t + v_{\alpha\mu} \cdot \mu_t + v_{\mu\alpha} \cdot \alpha_t + v_{\mu\mu} \cdot \mu_t = \frac{1}{4}v_{\alpha\alpha} - \frac{3}{4}v_{\alpha\mu} - v_{\mu\mu} \\ u_{tt} &= \frac{1}{4}v_{\alpha\alpha} \cdot \alpha_t + \frac{1}{4}v_{\alpha\mu} \cdot \mu_t - v_{\mu\alpha} \cdot \alpha_t - v_{\mu\mu} \cdot \mu_t = \frac{1}{16}v_{\alpha\alpha} - \frac{1}{2}v_{\alpha\mu} + v_{\mu\mu} \end{aligned}$$

$$\begin{aligned} u_{xx} - 3u_{xt} - 4u_{tt} = {} & 1v_{\alpha\alpha} + 2v_{\alpha\mu} + 1v_{\mu\mu} \\ & -\frac{3}{4}v_{\alpha\alpha} + \frac{9}{4}v_{\alpha\mu} + 3v_{\mu\mu} \\ & -\frac{1}{4}v_{\alpha\alpha} + 2v_{\alpha\mu} - 4v_{\mu\mu} \\ & = \frac{25}{4}v_{\alpha\mu} = 0 \iff \boxed{v_{\alpha\mu} = 0} \end{aligned}$$

Further alternative approach for transformation:

Matrix form: $(\nabla^T \boldsymbol{A}\, \nabla)u + (b^T\nabla)u + cu = h$.

Here the Matrx form of the PDE is

$$(\nabla^T A \nabla)u \;=\; \nabla^T \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & -4 \end{pmatrix} \nabla \cdot u = \mathbf{0}$$

With $S^T := \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & -1 \end{pmatrix}$ we obtain the following PDE for $\boldsymbol{v}$

$\nabla^T_{\alpha\mu} \S^T \boldsymbol{A}\, \S \nabla_{\alpha\mu}\, v = \mathbf{0}$.

where $\S^T \boldsymbol{A}\, \S = \begin{pmatrix} 1 & \frac{1}{4} \\ 1 & -1 \end{pmatrix} \begin{pmatrix} 1 & -\frac{3}{2} \\ -\frac{3}{2} & -4 \end{pmatrix} \begin{pmatrix} 1 & 1 \\ \frac{1}{4} & -1 \end{pmatrix} = \begin{pmatrix} 0 & \frac{20}{8} \\ \frac{20}{8} & 0 \end{pmatrix}$

Hence the PDE for v is $\boxed{v_{\alpha\mu} = 0}$

Solution of the transformed PDE:

From $(v_\alpha)_\mu = 0$ it follows that v_α does not depend on μ.

$$v(\alpha,\mu)_\alpha \;=\; \phi(\alpha) \overset{\int d\alpha}{\Longrightarrow} v(\alpha,\mu) \;=\; \Phi(\alpha) + \chi(\mu)$$

and

$$u(x,t) = v(\alpha,\mu) = \Phi(x + \frac{t}{4}) + \chi(x - t)$$

with sufficiently smooth functions Φ and χ.

From the initial data we obtain two conditions

$$u(x,0) = \Phi(x) + \chi(x) \overset{!}{=} 0 \quad \text{and}$$
$$u_t(x,0) = \frac{1}{4}\Phi'(x) - \chi'(x) \overset{!}{=} 2xe^{-x^2} \;\Rightarrow\; \frac{1}{4}\Phi(x) - \chi(x) \overset{!}{=} \int_{x_0}^{x} 2ze^{-z^2}\, dz \;=\; -e^{-z^2}\Big|_{x_0}^{x}.$$

By adding these two equations we have

$$\frac{5}{4}\Phi(x) = \; -e^{-x^2} + e^{-x_0^2}.$$

Subtracting four times the second equation from the first one, we obtain

$$5\chi(x) = 4e^{-x^2} - 4e^{-x_0^2}.$$

The solution to the initial value problem is therefore given by

$$u(x,t) = \;\Phi(x + \frac{t}{4}) + \chi(x - t) \;=\; -\frac{4}{5}e^{-(x+\frac{t}{4})^2} \;+\; \frac{4}{5}e^{-(x-t)^2}.$$