Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Work sheet 3

Exercise 1:

a) Determine the entropy solution $u(x,t)$ to the Burgers' equation $u_t + uu_x = 0$ for the initial values $u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 2 & 1 < x \end{cases}$

b) Given the following initial value problem for $u(x,t)$:

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} \frac{1}{2} & x \le 0, \\ 0 & 0 < x \le 1, \\ -2 & 1 < x\,. \end{cases}$$

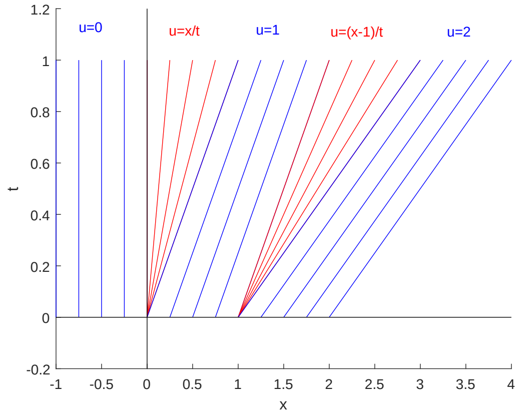
(i) Compute the weak solution for $t \in [0, \tilde{t}\,]$ with a sufficiently small $\tilde{t}$.

(ii) To what maximum t^* can the solution from i) be continued?

(iii) Provide the weak solution for $t > t^*$.

Solution sketch:

a) $u_t + uu_x = 0\,,\ u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 2 & x > 1 \end{cases} \longrightarrow$ Two rarefaction waves



$$u(x,t) = \begin{cases} 0 & x \le 0 \\ \frac{x}{t} & 0 \le x \le t \\ 1 & t \le x \le t+1 \\ \frac{x-1}{t} & t+1 \le x \le 2t+1 \\ 2 & x \ge 2t+1 \end{cases}$$

b) (i) At two jump points of the initial data, we introduce two shock waves. The jump condition requires:

$\dot{s}_1(t) = \dfrac{\frac{1}{2}+0}{2} = \frac{1}{4}$ and $\dot{s}_2(t) = \dfrac{0-2}{2} = -1\,.$

We obtain the shock fronts

$s_1(t) = \frac{1}{4}\,t$ and $s_2(t) = 1-t\,.$

For sufficiently small t

$$u(x,t) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{4}\,t, \\ 0 & \frac{1}{4}\,t < x \leq 1-t, \\ -2 & 1-t < x\,. \end{cases}$$

is a weak solution.

(ii) For t^* with

$\frac{1}{4}\,t^* = 1 - t^* \iff \frac{5}{4}\,t^* = 1 \iff t^* = \frac{4}{5}$

the shock fronts meet and the solution from a) becomes ambiguous.

(iii) For $t^* = \frac{4}{5}$ it holds $s_1(t) = s_2(t) = \frac{1}{5}$ and

$$u(x,\frac{4}{5}) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{5}\,, \\ -2 & x > \frac{1}{5}\,. \end{cases}$$

We add the new shock front

$$s_3(t) = \frac{1}{5} + \dot{s}_3(t-\frac{4}{5}) = \frac{1}{5} + \frac{\frac{1}{2}-2}{2}\,(t-\frac{4}{5})$$

and obtain for $t > \frac{4}{5}$

$$u(x,t) = \begin{cases} \frac{1}{2} & x \leq \frac{1}{5} - \frac{3}{4}\,(t-\frac{4}{5}) = \frac{4}{5} - \frac{3}{4}\,t. \\ -2 & x > \frac{4}{5} - \frac{3}{4}\,t. \end{cases}$$

Exercise 2:

Determine entropy solutions to the differential equation

$$u_t + (f(u))_x = 0$$

with the flow function $f(u) = \frac{(u-2)^4}{2}$ and initial conditions

a) $u(x,0) = \begin{cases} 2 & x \leq 0, \\ 1 & 0 < x, \end{cases}$ and **b)** $u(x,0) = \begin{cases} 1 & x \leq 0, \\ 2 & 0 < x. \end{cases}$

Note: Only solutions for the given initial values are required. You don't need to give solutions for general initial values!

Solution:

Using usual notation we have $f(u) = \frac{(u-2)^4}{2}$. **[1 point]**

On the characteristic curves it holds

$\dot{x}(t) = f'(u) = 2(u-2)^3$ and $\dot{u}(t) = 0$.

The characteristics are straight lines with the constant slope $2(u(x(0),0)-2)^3$.

In part a), an ambiguity of the solution obtained using the methods of characteristics arises immediately (i.e. already at $t = 0$). A shock front $s(t)$ must be introduced with $u_l = 2$ and $u_r = 1$ **[1 point]**

with

$$\dot{s}(t) = \frac{f(u_l) - f(u_r)}{u_l - u_r} = \frac{\frac{(2-2)^4}{2} - \frac{(1-2)^4}{2}}{2-1} = -\tfrac{1}{2} \quad \textbf{[1 point]}$$

We obtain

$$u(x,t) = \begin{cases} u_l = 2 & x < s(t) = -\frac{t}{2} \\ u_r = 1 & -\frac{t}{2} < x. \quad \textbf{[1 point]} \end{cases}$$

For part b) the method of characteristics gives

$$u(x,t) = \begin{cases} 1 & x \leq x_0 + f'(u_l)t = 0 + 2(1-2)^3 t = -2t\,, \\ \;? & -2t \leq x \leq 0\,, \\ 2 & x \geq x_0 + f'(u_r)t = 0 + 2(2-2)^3 t = 0\,. \end{cases}$$

A rarefaction wave must therefore be introduced. **[1 point]**

With

$f'(u) = 2(u-2)^3 = v \implies g(v) := (f')^{-1}(v) = \left(\frac{v}{2}\right)^{\frac{1}{3}} + 2$

we have the solution

$$u(x,t) = \begin{cases} 1 & x \leq -2t\,, \\ g(\frac{x}{t}) = \left(\frac{x}{2t}\right)^{\frac{1}{3}} + 2 & -2t \leq x \leq 0 \\ 2 & x \geq 0. \end{cases} \quad \textbf{[2 points]}$$

Exercise 3: (Only for the very fast students)

Physical processes described by smooth solutions to hyperbolic differential equations are generally reversible. If the solution is known at a certain time, one can use it to determine the solution at both later and earlier times.

Draw the characteristics for both initial value problems for the Burgers' equation $u_t + uu_x = 0$ with initial data

$$u_1(x,0) = \begin{cases} 1 & x \le 0, \\ 0 & x > 0. \end{cases}$$

and

$$u_2(x,0) = \begin{cases} 1 & x < -\frac{1}{4}, \\ \frac{1}{2} - 2x & -\frac{1}{4} \le x \le \frac{1}{4}, \\ 0 & x > \frac{1}{4}. \end{cases}$$

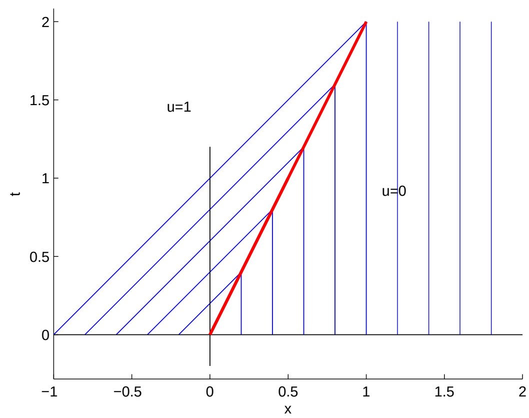
Determine the solution $u(x,1)$ for both initial value problems at time $t = 1$.

What do you conclude from your results regarding reversibility of non-smooth solutions to Burgers' equation?

Solution sketch to exercise 3: $u_t + uu_x = 0$

The initial data $u_1(x,0) = \begin{cases} 1 & x \le 0 \\ 0 & x > 0 \end{cases}$ already known from the lecture

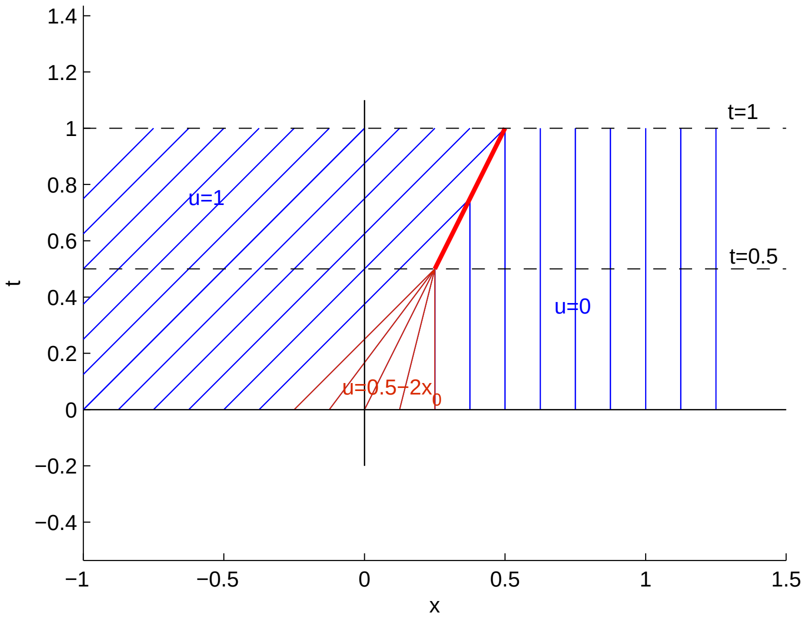
results in the following plot of characteristics



From the very beginning, a shock wave with the velocity $\frac{1}{2}$ occurs, and it holds

$$u_1(x,1) = \begin{cases} 1 & x \le \frac{1}{2} \\ 0 & x > \frac{1}{2} \end{cases}.$$

The initial data $u_2(x,0) = \begin{cases} 1 & x < -\dfrac{1}{4} \\ \dfrac{1}{2} - 2x & -\dfrac{1}{4} \leq x \leq \dfrac{1}{4} \\ 0 & x > \dfrac{1}{4} \end{cases}$ gives:



On the characteristics it holds $x - x_0 = u_0.t$.

For $-\frac{1}{4} \leq x_0 \leq \frac{1}{4}$ and $t = \frac{1}{2}$ so

$x(\frac{1}{2}) = x_0 + \left(\frac{1}{2} - 2x_0\right)\frac{1}{2} = \frac{1}{4}$

The characteristics that start at $t = 0$ between $x = -\frac{1}{4}$ and $x = \frac{1}{4}$ all intersect at the point $\left(\frac{1}{4}, \frac{1}{2}\right)$.

For $t = \frac{1}{2}$ it holds $u(x, \frac{1}{2}) = \begin{cases} 1 & x \leq \frac{1}{4} \\ 0 & x > \frac{1}{4} \end{cases}$. A shock wave is thus created with $x_0 = \dfrac{1}{4}$ and $u_l = 1$, $u_r = 0$. Hence for the shock front it holds

$$s(\tfrac{1}{2}) = \tfrac{1}{4}, \qquad \dot{s}(t) = \frac{f_l - f_r}{u_l - u_r} = \frac{\frac{u_l^2}{2} - \frac{u_r^2}{2}}{u_l - u_r} = \tfrac{1}{2}$$

$$\Longrightarrow s(t) = \tfrac{1}{4} + \tfrac{1}{2}(t - \tfrac{1}{2}) = \tfrac{t}{2}.$$

Hence for $t \geq \frac{1}{2}$ we have: $u_2(x,t) = \begin{cases} 1 & x \leq \frac{t}{2}, \\ 0 & x > \frac{t}{2} \end{cases}$

and then for $t = 1$ also: $$u_2(x,1) = \begin{cases} 1 & x \leq \frac{1}{2} \\ 0 & x > \frac{1}{2}\,. \end{cases}$$

Obviously, knowing the solution at time $t = 1$, one cannot reconstruct the solution at earlier times around $t = 0$.

Additionally (not required by the assignment):

For $t < 0.5$ it holds:

$u(x,t) = 1$ *for* $x \leq t - 0.25$,

$u(x,t) = 0$ *for* $x \geq 0.25$.

In between, i.e. for $t - 0.25 < x < 0.25$, *the following holds:*

$\frac{dx}{dt} = u, \qquad \frac{du}{dt} = 0.$ *i.e.* u *is constant along characteristics and*

$x = ut + c \iff c = x(0) = x - ut$

$\implies u(x,t) = u_0(x - ut) = \frac{1}{2} - 2(x - ut) \iff u \cdot (1 - 2t) = \frac{1}{2} - 2x$

So $u(x,t) = \dfrac{1 - 4x}{2 - 4t}$ *für* $t - 0.25 < x < 0.25,\ 0 < t < 0.5$.