Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Exercise sheet 2

Exercise 1: Old exam question

Given the initial value problem

$$u_t + 4t\, u_x = 3\,, \qquad x \in \mathbb{R},\, t \in \mathbb{R}^+,$$
$$u(x,0) = \sin(2x) \qquad x \in \mathbb{R}\,.$$

a) determine the equations of the characteristics and solve them,

b) compute the solution to the initial value problem for $u(x,t)$.

Solution:

a) With the method of characteristics one computes
$\frac{dx}{dt} = 4t \implies dx = 4tdt \implies x = 2t^2 + C_1$
$\frac{du}{dt} = 3 \implies du = 3dt \implies u = 3t + C_2$

b) With $C_1 = x - 2t^2$ and $C_2 = u - 3t$ we make an ansatz
$C_2 = f(C_1)$
and obtain
$u - 3t = f(x - 2t^2)$
and thus the general solution: $u(x,t) = 3t + f(x - 2t^2)$.
The initial condition requires:
$u(x,0) = 0 + f(x - 0) = f(x) \stackrel{!}{=} \sin(2x)$.
$u(x,t) = 3t + \sin(2x - 4t^2)$.

Exercise 2: (For the very fast students)

Solve the Cauchy problem

$$\begin{aligned} u_t - 4e^{-x}u_x &= 1 \qquad && x \in \mathbb{R}\,, t > 0\,, \\ u(x,0) &= x && x \in \mathbb{R}\,. \end{aligned}$$

Solution:

With t one computes as a parameter

$$\begin{array}{lcl} \dfrac{dx}{dt} = -4e^{-x} & \Longrightarrow & e^x\,dx \;=\; -4\,dt \\ \Longrightarrow e^x \;=\; -4t + c & \Longrightarrow & c = e^x + 4t \\ \dfrac{du}{dt} = 1 & \Longrightarrow & u(t) = t + d,\; d = u - t \\ \text{If } \Phi \text{ solvable by} & u - t\;: & \\ \Phi(e^x + 4t, u - t) = K & \Longrightarrow & u - t = f(e^x + 4t) \\ u(x,0) = f(e^x) = x & \Longrightarrow & f(x) = \ln(x) \\ u(x,t) = f(e^x + 4t) + t & \Longrightarrow & u(x,t) = \ln(e^x + 4t) + t \end{array}$$

Exercise 3: Old exam question

Given are the following differential equations for $u(x,t)$, $u : \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$

A) $u_t + 3u^3 u_x = 0,$

B) $u_t + 3x u_x = 0,$

C) $u_t + 3 u_x = 1\,.$

and the initial condition

$u(x,0) = u_0(x), \qquad x \in \mathbb{R}\,,$

where $u_0 : \mathbb{R} \to \mathbb{R}$ is a monotonically increasing and continuously differentiable function.

For which of the differential equations A), B), C) do the following statements i) and or ii) hold for the solution of the associated initial value problem?

i) The solution is constant along the characteristics?
ii) Are the characteristics straight lines?

Justify your answers.

Solution:

For A) it holds

$\frac{du}{dt} = 0 \Longrightarrow u$ is therefore constant along the characteristics.

On the other hand, the characteristics have the slope $\frac{dx}{dt} = 3u^3\,.$

The slope of the characteristics is therefore constant. These are straight lines.

For B) it holds $\frac{du}{dt} = 0 \Longrightarrow u$ is therefore constant along the characteristics.

The characteristics have the slope $\frac{dx}{dt} = 3x\,.$

The slope of the characteristics is not constant. They are not straight lines.

For C) it holds

$\frac{du}{dt} = 1 \Longrightarrow u$ is therefore not constant along the characteristics.

On the other hand, the characteristics have the slope $\frac{dx}{dt} = 3\,.$

The slope of the characteristics is therefore constant. They are straight lines.

Discussion: 24.04.23 - 28.04.23