Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Homework sheet 2

Exercise 1:

Determine the solutions to the following initial value problems for $t \in \mathbb{R}^+$, $x \in \mathbb{R}$.

a) $u_t + 3u_x = 0$ with $u(x,0) = u_0(x) = xe^{-x}$.

b) $2u_t + x^2 u_x = \frac{1}{u}$ with $u(x,0) = 2\sqrt{e^{-4x^2}}$.

Does there exist a solution for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}$?

If not, can the solution be continuously extended in the definition gaps (to be defined in the whole domain)?

Solution:

a) $u_t + 3u_x = 0$ with $u(x,0) = xe^{-x}$.

On a fixed characteristic $(t, x(t))$ we have:

$\dot{x}(t) = 3 \Longrightarrow x(t) = c + 3t$, $x(0) = c = x_0 = x - 3t$.

$\dot{u}(t) = 0 \Longrightarrow u$ is constant on the characteristic!

So

$$\Longrightarrow \begin{cases} u(x,t) &= u(x_0,0) = u(x-3t,0) = u_0(x-3t) \\ u(x,0) &= xe^{-x} \end{cases}$$

$$\Longrightarrow \quad u(x,t) = (x-3t)e^{-(x-3t)}$$

b) For $x = 0$ one gets an ordinary differential equation $2u_t = \frac{1}{u}$ with the solution $u(0,t) = \sqrt{t+C}$. Using initial value we get $C = 4$. For $x \neq 0$ we compute as follows

$$\begin{array}{llll} \frac{dx}{dt} = \frac{x^2}{2} & \frac{dx}{x^2} = \frac{dt}{2} & -\frac{1}{x} = \frac{t}{2} - C_1 & C_1 = \frac{t}{2} + \frac{1}{x} \\ \frac{du}{dt} = \frac{1}{2u} & 2u \cdot du = dt & u^2 = t + C_2 & C_2 = u^2 - t \end{array}$$

$$C_2 = f(C_1) \Longleftrightarrow u^2 - t = f(\frac{t}{2} + \frac{1}{x})$$

From the initial data follows

$$(u(x,0))^2 - 0 = 4\,e^{-4x^2} = f(\frac{1}{x})\,.$$

So

$$f(y) \;=\; 4\,e^{-4(1/y)^2} \implies u^2 = t + 4\exp\left(-4(\frac{t}{2}+\frac{1}{x})^{-2}\right)$$

$$u(x,t) = \sqrt{t + 4\exp\left(-4(\frac{(2x)^2}{(tx+2)^2}\right)} = \sqrt{t + 4e^{\frac{-16x^2}{(tx+2)^2}}}\,.$$

The soltuion is not defined for $x(t) = -2/t$. For every fixed $t \in \mathbb{R}^+$ it holds $x^2 = 4/t^2 \neq 0$ and

$$\lim_{x\to -2/t} \frac{-16x^2}{(tx+2)^2} = -\infty \Longrightarrow \lim_{x\to -2/t} e^{\frac{-16x^2}{(tx+2)^2}} = 0$$

and hence

$$\lim_{x\to -2/t} u(x,t) \;=\; \sqrt{t}\,.$$

Exercise 2:

A simple traffic flow model:

We consider a one-dimensional flow of vehicles along an infinitely long, single-lane road. In a so-called macroscopic model, one does not consider individual vehicles, but the total flow of vehicles. For this purpose, we introduce the following quantities:

$$\begin{aligned} u(x,t) &= \text{(length-)density of the vehicles at the point } x \text{ at the time } t \\ &= \text{vehicles/unit length at point } x \text{ at the time } t \\ v(x,t) &= \text{ speed at the point } x \text{ at the time } t, \\ q(x,t) &= u(x,t)\cdot v(x,t) = \text{ flow} \\ &= \text{ amount of vehicles passing the point } x \text{ at the time } t \text{ per unit time} \end{aligned}$$

a) Assume that there are no entrances or exits, no vehicles are disappearing, and no new vehicles are appearing. Let $N(t,a,\Delta a) :=$ number of vehicles on a space interval $[a, a+\Delta a]$ at the time t.

   Then on the one hand it holds that

   $$N(t,a,\Delta a) \;=\; \int_a^{a+\Delta a} u(x,t)\,dx$$

   and on the other hand it also holds

   $$N(t,a,\Delta a) - N(t_0,a,\Delta a) \;=\; \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau)d\tau\,.$$

   Derive from this the so-called conservation equation for the mass (number of vehicles)

   $$u_t + q_x = 0.$$

   Hints on how to proceed:

   - Derive both formulas for N with respect to t. Please note that for the derivation of parameter-dependent integrals with sufficiently smooth f holds the **Leibniz rule**:

   $$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt \,+\, b'(x)\,f(x,b(x)) \,-\, a'(x)\,f(x,a(x))}$$

   - Divide by Δa.
   - Consider the limit $\Delta a \to 0$.

b) Additionally assume that the velocity depends only on the density: $v = v(u)$. Show that in this case the equation

   $$\frac{\partial u}{\partial t} + \frac{dq}{du}\cdot\frac{\partial u}{\partial x} \;=\; 0$$

   describes the conservation of mass.

c) We now assume in a first simple model that the speed increases in inverse proportion to the density and that the density is positive.

$$v(x,t) \,=\, c + \frac{k}{u(x,t)}$$

What is the continuity equation (=conservation equation for the mass)?

Solution:

a) On the one hand, it holds $\quad N(t) \,=\, \int_a^{a+\Delta a} u(x,t)\,dx$

and on the other hand $\quad N(t) - N(t_0) \,=\, \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau) d\tau\,.$

Differentiating with respect to t gives

$$\frac{\partial}{\partial t} N(t) \,=\, \frac{\partial}{\partial t}\int_a^{a+\Delta a} u(x,t)\,dx = \, q(a,t) - q(a+\Delta a,t)$$

Letting Δa to zero, and with sufficient smoothness of the functions, we have

$$\lim_{\Delta a\to 0} \frac{1}{\Delta a}\int_a^{a+\Delta a} \frac{\partial}{\partial t} u(x,t)\,dx = \lim_{\Delta a\to 0} -\frac{q(a+\Delta a,t) - q(a,t)}{\Delta a}$$

$$\Longrightarrow \frac{\partial}{\partial t} u(a,t) = -\frac{\partial}{\partial a} q(a,t).$$

Since these considerations hold at every point, we have the continuity equation

$u_t + q_x = 0\,.$

b) Actually is straightforward, since in this case we have $q(x,t) = u(x,t)\cdot v(u(x,t))$. The flow q is therefore a function of $u(x,t)$. The assertion then follows from the chain rule.

In more details:

With $q(x,t) = u(x,t)\cdot v(u(x,t))$ we have

$$\frac{dq}{du}\cdot\frac{\partial u}{\partial x} \,=\, \frac{d}{du}\,(u\cdot v(u))\cdot u_x = \,(v(u) + u\cdot v_u)\cdot u_x$$

and on the other hand it holds

$$\frac{\partial}{\partial x} q(x,t) \,=\, \frac{\partial}{\partial x}\,(u(x,t)\cdot v(u(x,t))) \,= u_x\cdot v(u) + u\cdot v_u\cdot u_x\,.$$

c)

$$v(x,t) \;=\; c + \frac{k}{u(x,t)} \qquad q(x,t) = \; c \cdot u(x,t) \; + \; k$$

From the continuity equation from part b) we have

$$\frac{\partial u}{\partial t} + \frac{dq}{du} \cdot \frac{\partial u}{\partial x} = \frac{\partial u}{\partial t} + c \cdot \frac{\partial u}{\partial x} = 0$$

The linear transport equation is thus obtained.

For $c = 3$ the equation is solved in Exercise 1a).

Note : This is a very simple, linearized model. For example, it allows for any density and any speed. A somewhat more realistic problem would already produce shock and rarefaction waves (see later exercises).

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