Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Work sheet 1

Exercise 1: (Repetition of DGL I)

a) Let λ be any fixed real number. Determine a real representation of the general solution to the differential equation

$$y''(t) - \lambda\, y(t) \,=\, 0\,.$$

b) Let L be another fixed positive real number. Determine all solutions to the boundary value problem

$$y''(t) - \lambda\, y(t) \,=\, 0 \quad y(0) \,=\, y(L) \,= 0\,.$$

For which $\lambda \in \mathbb{R}$ does the boundary value problem have nontrivial solutions?

The λ-values for which there exist non-trivial solutions (i.e. solutions that are not constantly equal to zero) are called eigenvalues of the problem. The corresponding solutions are called eigenfunctions.

Remark: *The solutions to this eigenvalue problem will be needed again and again during the semester!*

Solution hints for the exercise 1:

a) Following DGL I, we calculate the characteristic polynomial : $\mu^2 - \lambda \,=\, 0$ with the zeros

$$\mu_{1,2} \,=\, \pm\sqrt{\lambda} \quad \Longrightarrow y(t) = \begin{cases} c_1\, e^{\sqrt{\lambda}t} \,+\, c_2\, e^{-\sqrt{\lambda}t} & \lambda \,>\, 0\,, \\ c_1 \,+\, c_2 t & \lambda \,= 0\,, \\ c_1\, \cos(\sqrt{-\lambda}t) \,+\, c_2\, \sin(\sqrt{-\lambda}t) & \lambda \,<\, 0\,. \end{cases}$$

b) For case $\lambda \,>\, 0,$ from the boundary value for $t \,=\, 0$ it follows immediately that $c_2 = -c_1$. The boundary value at L yields:

$$c_1 \left(e^{\sqrt{\lambda}L} \,-\, e^{-\sqrt{\lambda}L}\right) \,=\, 0 \Longrightarrow c_1 \left(e^{2\sqrt{\lambda}L} \,-\, 1\right) \,=\, 0 \Longrightarrow c_1 = 0$$

In this case there exist only the trivial solution $y(t) = 0$

For case $\lambda \,=\, 0,$ the solution is a linear function. The only linear function that exists in $t = 0$ and also disappears in $t = L > 0$ is again the trivial solution.

For case $\lambda \,<\, 0,$ from the boundary value for $t \,=\, 0$ it follows immediately that $c_1 = 0$. The boundary value in L yields:

$$c_2\, \sin(\sqrt{-\lambda}L) = 0 \Longrightarrow c_2 = 0 \,\vee\, \sqrt{-\lambda}L = k\pi$$

Non-trivial solutions only exist for $\lambda \,=\, -\left(\frac{k\pi}{L}\right)^2.$

Exercise 2) (Repetition Analysis II)

Given $$f(t) = \begin{cases} 4t & t \in [0, \frac{1}{2}] \\ 4 - 4t & t \in [\frac{1}{2}, 1] \\ 0 & t \in [1, 2] \end{cases}$$

a) make a sketch of the direct 2-periodic continuation of f, and even and odd 4-periodic continuation of f

b) compute the real Fourier series of the odd 4–periodic continuation of f,

c) compute the real Fourier series of the even, 4–periodic extension of f.

Recall:

Let $f : \mathbb{R} \to \mathbb{R}$ integrable and **periodic** with period $T > 0$, i.e.

$$f(t+T) = f(t), \qquad \forall t \in \mathbb{R}$$

Define the angular frequency $\omega = \dfrac{2\pi}{T}$ and denote by

$T_n := \left\{ g : \mathbb{R} \to \mathbb{R}, g(t) = \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(k\omega t) + b_k \sin(k\omega t)), \; a_k, b_k \in \mathbb{R}, \right\}$ the space of all $T-$ periodic **trigonometric polynomials of degree n** with the

inner product: $< f, g > := \frac{2}{T} \int_0^T f(t) \cdot g(t) \, dt$.

and the norm $\| g\| := \sqrt{\frac{2}{T} \int_0^T (g(t))^2 \, dt} = \sqrt{< g, g >}$.

Then the functions $\left\{ \frac{1}{\sqrt{2}}, \cos(k\omega t), \sin(k\omega t) : \quad k \in \mathbb{N} \right\}$
are an orthogonal system and **truncated Fourier series of** f

$$f_n(x) := \frac{a_0}{2} + \sum_{k=1}^{n} (a_k \cos(k\omega t) + b_k \sin(k\omega t))$$

with

$$a_k := \frac{2}{T} \int_0^T f(t) \cos(k\omega t) \, dt, \qquad k \in \mathbb{N}_0,$$
$$b_k := \frac{2}{T} \int_0^T f(t) \sin(k\omega t) \, dt, \qquad k \in \mathbb{N}.$$

is the best approximation for f from T_n, i.e.

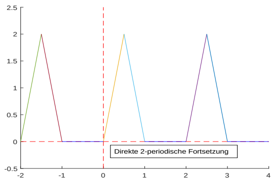
$$\| f - f_n \| < \| f - g \| \quad \forall g \in T_n.$$

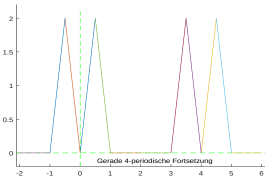
If **f even** then $b_k = 0$ and $a_k = \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t) \cos(k\omega t) \, dt \qquad k \in \mathbb{N}_0$.

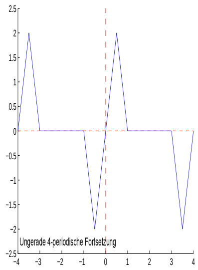
If **f is odd** then $a_k = 0$ and $b_k = \dfrac{4}{T} \displaystyle\int_0^{T/2} f(t) \sin(k\omega t) \, dt \qquad k \in \mathbb{N}$.

Solution sketch:

a)







b) $a_k = 0$ because function is odd!

$$T = 4, \qquad \omega = \frac{2\pi}{T} = \frac{\pi}{2}$$

$$b_k = \frac{4}{T}\int_0^{T/2} f(t)\sin(k\omega t)\,dt = \int_0^2 f(t)\sin(\frac{k\pi}{2}t)dt$$

$$= \int_0^{1/2} 4t\sin(\frac{k\pi}{2}t)dt \;+\; \int_{1/2}^{1}(4-4t)\sin(\frac{k\pi}{2}t)dt$$

$$= 4\left[t\frac{-\cos\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_0^{\frac{1}{2}} + 4\int_0^{\frac{1}{2}} \frac{\cos\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}dt$$

$$+4\left[(1-t)\frac{-\cos(\frac{k\pi}{2}t)}{\frac{k\pi}{2}}\right]_{\frac{1}{2}}^{1} + \int_{\frac{1}{2}}^{1}(-4)\frac{\cos(\frac{k\pi}{2}t)}{\frac{k\pi}{2}}dt$$

$$= \frac{-8}{k\pi}\left(\tfrac{1}{2}\cos\left(\tfrac{k\pi}{4}\right)\right) + \frac{8}{k\pi}\left[\frac{\sin\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_0^{\frac{1}{2}} + \frac{8}{k\pi}\left(\tfrac{1}{2}\cos\left(\tfrac{k\pi}{4}\right)\right) - \frac{8}{k\pi}\left[\frac{\sin\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_{\frac{1}{2}}^1$$

$$= \frac{16}{(k\pi)^2}\sin(\tfrac{k\pi}{4}) - \frac{16}{(k\pi)^2}\left(\sin(\tfrac{k\pi}{2}) - \sin(\tfrac{k\pi}{4})\right) = \frac{16}{(k\pi)^2}\left(2\sin(\tfrac{k\pi}{4}) - \sin(\tfrac{k\pi}{2})\right)$$

Since f is continuously and piecewise continuously differentiable, the Fourier series converges to f:

$$f(t) = \sum_{k=1}^{\infty} \frac{16}{(k\pi)^2}\left(2\sin(\frac{k\pi}{4}) - \sin(\frac{k\pi}{2})\right)\sin(\frac{k\pi}{2}t)$$

c) $b_k = 0$ since function is even!

$T = 4, \qquad \omega = \frac{2\pi}{T} = \frac{\pi}{2}$

$a_0 = \frac{4}{T}\int_0^{T/2} f(t)\,dt = \int_0^2 f(t)dt = \int_0^{1/2} 4tdt + \int_{1/2}^1 (4-4t)dt = \frac{1}{2} + 2 - 2 + \frac{1}{2} = 1\,.$

For $k \in \mathbb{N}$ one computes as follows

$a_k = \frac{4}{T}\int_0^{T/2} f(t)\cos(k\omega t)\,dt = \int_0^2 f(t)\cos(\frac{k\pi}{2}t)dt$

$$= \int_0^{1/2} 4t\cos(\frac{k\pi}{2}t)dt + \int_{1/2}^1 (4-4t)\cos(\frac{k\pi}{2}t)dt$$

$$= 4\left[t\frac{\sin\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_0^{\frac{1}{2}} - 4\int_0^{\frac{1}{2}} \frac{\sin\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}dt + 4\left[(1-t)\frac{\sin(\frac{k\pi}{2}t)}{\frac{k\pi}{2}}\right]_{\frac{1}{2}}^1 - \int_{\frac{1}{2}}^1 (-4)\frac{\sin(\frac{k\pi}{2}t)}{\frac{k\pi}{2}}dt$$

$$= \frac{4}{k\pi}\left(\sin\left(\tfrac{k\pi}{4}\right)\right) - \frac{8}{k\pi}\left[\frac{-\cos\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_0^{\frac{1}{2}} - \frac{4}{k\pi}\left(\sin\left(\tfrac{k\pi}{4}\right)\right) - \frac{8}{k\pi}\left[\frac{\cos\left(\frac{k\pi}{2}t\right)}{\frac{k\pi}{2}}\right]_{\frac{1}{2}}^1$$

$$= \frac{16}{k^2\pi^2}\left[2\cos\left(\tfrac{k\pi}{4}\right) - \cos\left(\tfrac{k\pi}{2}\right) - 1\right]$$

Since f is continuously, piecewise continuously differentiable, the Fourier series converges to f:

$$f(t) = \frac{1}{2} + \sum_{k=1}^{\infty} \left\{ \frac{16}{k^2\pi^2} \left[2\cos\left(\frac{k\pi}{4}\right) - \cos\left(\frac{k\pi}{2}\right) - 1 \right] \right\} \cos(\frac{k\pi}{2}t)$$