Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Homework sheet 1

Exercise 1: (Repetition Analysis II)

For the derivation of parameter-dependent integrals for sufficiently smooth f holds the **Leibniz–Rule** :

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))}$$

Find the derivative of the function $F(x)$ defined as

$$F(x) := \int_{-x}^{x^2} e^{xt}\,dt$$

and compute $\lim\limits_{x\to 0} F'(x)$.

Solution to Exercise 1:

$$F(x) = \int_{-x}^{x^2} e^{xt}\,dt\,, \qquad b(x) := x^2,\ a(x) := -x,\ f(t,x) := e^{xt}$$

$$\begin{aligned} b'(x) &= 2x & a'(x) &= -1 \\ f(b(x),x) &= e^{x^3} & f(a(x),x) &= e^{-x^2} \end{aligned}$$

$$\begin{aligned} F'(x) &= \int_{a(x)}^{b(x)} \frac{\partial}{\partial x} f(t,x)\,dt + b'(x)f(b(x),x) - a'(x)f(a(x),x) \\ &= \int_{-x}^{x^2} te^{xt}\,dt + 2x\,e^{x^3} + e^{-x^2} \\ &= \left[\frac{t}{x}e^{tx}\right]_{-x}^{x^2} - \int_{-x}^{x^2} \frac{1}{x}e^{xt}\,dt + 2x\,e^{x^3} + e^{-x^2} \\ &= 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left[e^{tx}\right]_{-x}^{x^2} = 3x\,e^{x^3} + 2e^{-x^2} - \frac{1}{x^2}\left(e^{x^3} - e^{-x^2}\right) \end{aligned}$$

Substitution/ L'Hospital gives:

$$F'(0) = 0 + 2 - \lim_{x \to 0} \frac{1}{x^2}(e^{x^3} - e^{-x^2}) = 2 - \lim_{x \to 0} \frac{3x^2 e^{x^3} + 2xe^{-x^2}}{2x} = 2 - 1 = 1\,.$$

Exercise 2: (Repetition of Analysis II)

Determine the appropriate real Fourier series for the following functions:

a) Odd $2L-$ periodic continuation of
$f : [0,1[\to \mathbb{R}, \quad f(x) = \sin(4\pi x) + 2\sin(6\pi x) \quad L = 1\,.$

b) Even $2L-$ periodic continuation of
$f : [-\frac{\pi}{4}, \frac{5\pi}{4}[\to \mathbb{R}, \quad L = \pi$ with

$$f(t) = \begin{cases} 2, & -\frac{\pi}{4} \le t < \frac{\pi}{4}, \\ 0, & \frac{\pi}{4} \le t < \frac{3\pi}{4}, \\ 2, & \frac{3\pi}{4} \le t < \frac{5\pi}{4}. \end{cases}$$

Remark: *For DGL II you will need to know how to calculate Fourier series. Please repeat if necessary!*

Solution hint to Exercise 2:

a) Since the function $f(x)$ is continued oddly, a Fourier sine series is used. Since $2L$ is a period of the function, one chooses $2L-$ periodic sine functions. So we define a series in the form

$$F(x) = \sum_{k=1}^{\infty} b_k \sin\left(k\frac{2\pi}{2L}\,x\right)$$

$$L = 1 \implies F(x) = \sum_{k=1}^{\infty} b_k \sin(k\pi x)$$

Due to orthogonality relations between the $\sin(k\pi x)$ and $\sin(l\pi x)$ (see Mathe II) and by assuming that the Fourier series is as good as possible approximation of f, we have

$$b_4 = 1\,, \quad b_6 = 2\,, \quad b_k = 0 \quad \text{otherwise.}$$

b) Since function $f(t)$ is continued evenly, a Fourier cosine series is used. Since $2L$ is a period of the function, one chooses $2L-$ periodic cosine functions. In general we define the series as

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{2\pi}{2L}\,t\right).$$

In our special case, we have that the continuation is even $\pi-$ periodic. So we can write the series in the following form

$$F(t) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos\left(k\frac{2\pi}{\pi}\,t\right).$$

Following Analysis II, and since $T = \pi$, we have for the coefficients

$$\begin{aligned} a_k &= \frac{4}{\pi}\int_0^{\frac{T}{2}} f(t)\cos(k\omega t)\,dt \\ &= \frac{4}{\pi}\int_0^{\frac{\pi}{4}} 2\cdot\cos\left(k\frac{2\pi}{\pi}t\right)dt + \frac{4}{\pi}\int_{\frac{\pi}{4}}^{\frac{\pi}{2}} 0\cdot\cos\left(k\frac{2\pi}{\pi}t\right)dt \\ &= \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\cos(2kt)dt \end{aligned}$$

For $k = 0$ it holds

$$a_0 = \frac{8}{\pi}\int_0^{\frac{\pi}{4}} 1dt = \frac{8}{\pi}[t]_0^{\frac{\pi}{4}} = 2$$

and for $k > 0$ we obtain

$$a_k = \frac{8}{\pi}\int_0^{\frac{\pi}{4}}\cos(2kt)dt = \frac{4}{\pi}\left[\frac{1}{k}\sin(2kt)\right]_0^{\frac{\pi}{4}} = \frac{4}{\pi k}\sin\left(\frac{k\pi}{2}\right).$$

Hence

$$a_k = \begin{cases} 2 & k = 0 \\ 0 & k = 2m,\ m \in \mathbb{N} \\ \frac{4(-1)^m}{\pi(2m+1)} & k = 2m+1,\ m \in \mathbb{N}_0 \end{cases}$$

so

$$a_0 = 2 \qquad a_1 = \frac{4}{\pi} \qquad a_3 = -\frac{4}{3\pi} \qquad a_5 = \frac{4}{5\pi}\ \cdots$$

The first four non-vanishing summands of the Fourier series are e.g.

$$1 + \frac{4}{\pi}\cos(2t) - \frac{4}{3\pi}\cos(6t) + \frac{4}{5\pi}\cos(10t).$$

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