Fachbereich Mathematik der Universität Hamburg **SoSe 2023**
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Differential Equations II for Engineering Students

Work sheet 7

Exercise 1:

a) Given the initial boundary value problem

$$\begin{aligned} u_{tt} - 4u_{xx} &= -x \cdot \sin(t) && \text{for } x \in (0,1), t > 0, \\ u(x,0) &= 1 - x + 4\sin(2\pi x) && \text{for } x \in [0,1], \\ u_t(x,0) &= x + 3\sin(6\pi x) && \text{for } x \in [0,1], \\ u(0,t) = 1, \quad u(1,t) &= \sin(t) && \text{for } t > 0. \end{aligned}$$

Transform the problem using suitable homogenization of the boundary data into an initial boundary value problem with homogeneous boundary data.

b) Solve the following initial boundary value problem

$$\begin{aligned} v_{tt} - 4v_{xx} &= 0 && \text{for } x \in (0,1), t > 0, \\ v(x,0) &= 4\sin(2\pi x) && \text{for } x \in [0,1], \\ v_t(x,0) &= 3\sin(6\pi x) && \text{for } x \in [0,1], \\ v(0,t) = 0, \quad v(1,t) &= 0 && \text{for } t > 0. \end{aligned}$$

Exercise 2:

From lecture classes you know d'Alembert's formula

$$\hat{u}(x,t) = \frac{1}{2}\left(f(x+ct) + f(x-ct)\right) + \frac{1}{2c}\int_{x-ct}^{x+ct} g(\alpha)\,d\alpha$$

for the solution of the initial value problem for the (homogeneous) wave equation

$$\hat{u}_{tt} - c^2\hat{u}_{xx} = 0,\ \hat{u}(x,0) = f(x),\ \hat{u}_t(x,0) = g(x),\ x \in \mathbb{R},\ c > 0.$$

a) **(Just for the really quick participants)** Show that the function

$$\tilde{u}(x,t) = \frac{1}{2c}\int_0^t \int_{x+c(\tau-t)}^{x-c(\tau-t)} h(\omega,\tau)\,\mathrm{d}\omega\mathrm{d}\tau$$

solves the following inhomogeneous initial value problem.

$$\tilde{u}_{tt} - c^2\tilde{u}_{xx} = h(x,t) \qquad \tilde{u}(x,0) = \tilde{u}_t(x,0) = 0.$$

Hint: Leibniz formula for the derivation of parameter-dependent integrals (Sheet 1H):

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt \;=\; \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt \;+\; b'(x)\,f(x,b(x)) \;-\; a'(x)\,f(x,a(x))}$$

b) Solve the initial value problem

$$\begin{aligned} u_{tt} \;-\; 4u_{xx} \;&=\; -4x, && x \in \mathbb{R},\; t > 0 \\ u(x,0) \;&=\; 1, && x \in \mathbb{R}, \\ u_t(x,0) \;&=\; \cos(x), && x \in \mathbb{R} \end{aligned} \tag{1}$$

(i) Compute a solution $\hat{u}$ of the initial value problem

$$\begin{aligned} \hat{u}_{tt} \;-\; 4\hat{u}_{xx} \;&=\; 0, \quad x \in \mathbb{R},\; t > 0 \\ \hat{u}(x,0) \;&=\; 1,\; x \in \mathbb{R}, \\ \hat{u}_t(x,0) \;&=\; \cos(x),\; x \in \mathbb{R}. \end{aligned}$$

(ii) Compute a solution $\tilde{u}$ of the initial value problem using the result from part a).

$$\begin{aligned} \tilde{u}_{tt} \;-\; 4\tilde{u}_{xx} \;&=\; -4x, && x \in \mathbb{R},\; t > 0 \\ \tilde{u}(x,0) \;&=\; 0,\; x \in \mathbb{R}, && \tilde{u}_t(x,0) \;=\; 0,\; x \in \mathbb{R} \end{aligned}$$

(iii) By inserting u into the differential equation and checking the initial values, show that $u = \tilde{u} + \hat{u}$ solves the initial value problem (1).

Discussion: 10.07 - 14.07.2023