Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Homework sheet 5

Exercise 1:

a) We are looking for a solution of the Laplace equation $\Delta v(x,y) = 0$ in a rotationally symmetric area, for example in a circle. The area can then be better described using polar coordinates. This is done as follows

$x = r\cos\phi\,,\ y = r\sin\phi\,$, and

$v(x(r,\phi), y(r,\phi)) = u(r,\phi)$.

Show that for $r \neq 0$ the following equivalence holds:

$$r^2 u_{rr} + r u_r + u_{\varphi\varphi} = 0 \iff r^2\left(v_{xx} + v_{yy}\right) = 0\,.$$

b) Find a solution to the following boundary value problem:

$$\begin{aligned} \Delta(v) &= 0 \quad \text{für } 1 < x^2 + y^2 < 4, \\ v(x,y) &= 1 \quad \text{auf } x^2 + y^2 = 1, \\ v(x,y) &= 2 \quad \text{auf } x^2 + y^2 = 4. \end{aligned}$$

Hint: Use polar coordinates. The boundary data are independent of ϕ. So try the ansatz

$$v(x,y) = u(r,\phi) = w(r).$$

Exercise 2:

a) Show that through $\quad a_k = 0,\ \forall k \in \mathbb{N}_0, \quad \beta_k = \begin{cases} 0 & \text{for } k \in \mathbb{N} \text{ even,} \\ -\dfrac{8}{(k\pi)^3} & \text{for } k \in \mathbb{N} \text{ odd} \end{cases}$

the Fourier coefficients of the Fourier series

$$\frac{a_0}{2} + \sum_{k=1}^{\infty} \left(a_k \cos(k\pi y) + \beta_k \sin(k\pi y)\right)$$

of the odd, 2–periodic continuation of

$$g(y) = y^2 - y,\ 0 \le y \le 1$$

are given.

b) Determine with the help of an appropriate product ansatz and by using a) the solution of the following boundary value problem

$$\begin{aligned}
&\Delta u(x,y) = 0 && x \in (0,1),\ y \in (0,1),\\
&u(x,0) = 0 && x \in [0,1]\,,\\
&u(x,1) = 0 && x \in [0,1]\,,\\
&u(0,y) = g(y) = y^2 - y && y \in [0,1]\,,\\
&u(1,y) = 0 && y \in [0,1]\,.
\end{aligned}$$