**Fachbereich Mathematik der Universität Hamburg** **SuSe 2023**
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# Differential Equations II for Engineering Students

## Work sheet 3

**Exercise 1:**

a) Determine the entropy solution $u(x,t)$ to the Burgers' equation $u_t + uu_x = 0$ for the initial values $u(x,0) = \begin{cases} 0 & x < 0 \\ 1 & 0 \le x \le 1 \\ 2 & 1 < x \end{cases}$

b) Given the following initial value problem for $u(x,t)$:

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} \frac{1}{2} & x \le 0, \\ 0 & 0 < x \le 1, \\ -2 & 1 < x\,. \end{cases}$$

(i) Compute the weak solution for $t \in [0, \tilde{t}\,]$ with a sufficiently small $\tilde{t}$.

(ii) To what maximum $t^*$ can the solution from i) be continued?

(iii) Provide the weak solution for $t > t^*$.

**Exercise 2:**

Determine entropy solutions to the differential equation

$$u_t + (f(u))_x = 0$$

with the flow function $f(u) = \frac{(u-2)^4}{2}$ and initial conditions

**a)** $u(x,0) = \begin{cases} 2 & x \le 0, \\ 1 & 0 < x, \end{cases}$ and **b)** $u(x,0) = \begin{cases} 1 & x \le 0, \\ 2 & 0 < x. \end{cases}$

Note: Only solutions for the given initial values are required. You don't need to give solutions for general initial values!

**Exercise 3:** (Only for the very fast students)

Physical processes described by smooth solutions to hyperbolic differential equations are generally reversible. If the solution is known at a certain time, one can use it to determine the solution at both later and earlier times.

Draw the characteristics for both initial value problems for the Burgers' equation $u_t + uu_x = 0$ with initial data

$$u_1(x,0) = \begin{cases} 1 & x \leq 0, \\ 0 & x > 0. \end{cases}$$

and

$$u_2(x,0) = \begin{cases} 1 & x < -\frac{1}{4}, \\ \frac{1}{2} - 2x & -\frac{1}{4} \leq x \leq \frac{1}{4}, \\ 0 & x > \frac{1}{4}. \end{cases}$$

Determine the solution $u(x,1)$ for both initial value problems at time $t = 1$.

What do you conclude from your results regarding reversibility of non-smooth solutions to Burgers' equation?