Fachbereich Mathematik der Universität Hamburg **SuSe 2023**
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Differential Equations II for Engineering Students

Homework sheet 2

Exercise 1:

Determine the solutions to the following initial value problems for $t \in \mathbb{R}^+$, $x \in \mathbb{R}$.

a) $u_t + 3u_x = 0$ with $u(x,0) = u_0(x) = xe^{-x}$.

b) $2u_t + x^2 u_x = \dfrac{1}{u}$ with $u(x,0) = 2\sqrt{e^{-4x^2}}$.

Does there exist a solution for all $t \in \mathbb{R}^+$, $x \in \mathbb{R}$?

If not, can the solution be continuously extended in the definition gaps (to be defined in the whole domain)?

Exercise 2:

A simple traffic flow model:

We consider a one-dimensional flow of vehicles along an infinitely long, single-lane road. In a so-called macroscopic model, one does not consider individual vehicles, but the total flow of vehicles. For this purpose, we introduce the following quantities:

$u(x,t)$ = (length-)density of the vehicles at the point x at the time t
= vehicles/unit length at point x at the time t
$v(x,t)$ = speed at the point x at the time t,
$q(x,t)$ = $u(x,t) \cdot v(x,t)$ = flow
= amount of vehicles passing the point x at the time t per unit time

a) Assume that there are no entrances or exits, no vehicles are disappearing, and no new vehicles are appearing. Let $N(t,a,\Delta a) :=$ number of vehicles on a space interval $[a, a+\Delta a]$ at the time t.

Then on the one hand it holds that

$$N(t,a,\Delta a) = \int_a^{a+\Delta a} u(x,t)\,dx$$

and on the other hand it also holds

$$N(t,a,\Delta a) - N(t_0,a,\Delta a) = \int_{t_0}^{t} q(a,\tau) - q(a+\Delta a,\tau)d\tau\,.$$

Derive from this the so-called conservation equation for the mass (number of vehicles)

$$u_t + q_x = 0.$$

Hints on how to proceed:

- Derive both formulas for N with respect to t. Please note that for the derivation of parameter-dependent integrals with sufficiently smooth f holds the **Leibniz rule**:

$$\boxed{\frac{d}{dx}\int_{a(x)}^{b(x)} f(x,t)\,dt = \int_{a(x)}^{b(x)} \frac{d}{dx} f(x,t)dt + b'(x)\,f(x,b(x)) - a'(x)\,f(x,a(x))}$$

- Divide by Δa.
- Consider the limit $\Delta a \to 0$.

b) Additionally assume that the velocity depends only on the density: $v = v(u)$. Show that in this case the equation

$$\frac{\partial u}{\partial t} + \frac{dq}{du} \cdot \frac{\partial u}{\partial x} = 0$$

describes the conservation of mass.

c) We now assume in a first simple model that the speed increases in inverse proportion to the density and that the density is positive.

$$v(x,t) = c + \frac{k}{u(x,t)}$$

What is the continuity equation (=conservation equation for the mass)?

Note : This is a very simple, linearized model. For example, it allows for any density and any speed. A somewhat more realistic problem would already produce shock and rarefaction waves (see later exercises).

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