Mathematics Department
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WiSe 2023/24

Exam Differential Equations II

04. March 2024

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Task no.	Points	Evaluater
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Exercise 1: [7 points]

Consider the following initial value problem for $u(x,t)$:

$$u_t + u \cdot u_x = 0, \qquad x \in \mathbb{R},\ t \in \mathbb{R}^+$$

$$u(x,0) = \begin{cases} 4 & x \leq -1, \\ 0 & -1 < x \leq 0, \\ -4 & 0 < x\,. \end{cases}$$

a) Determine the entropy solution for $t \in [0, t^*)$ with a sufficiently small t^*.

b) Up to which t^* can the solution from a) be continued at most?

c) Determine the entropy solution for $t > t^*$.

Solution:

a) At two discontinuities of the initial data, we introduce two shock waves.

The jump condition requires:

$\dot{s}_1(t) = \dfrac{4+0}{2} = 2$ and $\dot{s}_2(t) = \dfrac{0-4}{2} = -2\,.$

We get shock fronts

$s_1(t) = -1 + 2t$ and $s_2(t) = -2t\,.$

For sufficiently small t we have

$$u(x,t) = \begin{cases} 4 & x \leq -1+2t, \\ 0 & -1+2t < x \leq -2t, \\ -4 & -2t < x\,. \end{cases} \qquad \textbf{(3 points)}$$

is a weak solution.

b) For t^* with

$-1 + 2t^* = -2t^* \iff 4t^* = 1 \iff t^* = \frac{1}{4}$ **(1 point)**

the shock fronts meet and the solution from a) becomes ambiguous.

c) For $t^* = \frac{1}{4}$ it holds $s_1(t) = s_2(t) = -\frac{1}{2}$ and

$$u(x, \frac{1}{4}) = \begin{cases} 4 & x \leq -\frac{1}{2}, \\ -4 & x > -\frac{1}{2}\,. \end{cases}$$

We create a new shock front s_3 with $\dot{s}_3(t) = \frac{4+(-4)}{2} = 0\,.$

$$s_3(t) = -\frac{1}{2} + \dot{s}_3(t)(t - \frac{1}{4}) = -\frac{1}{2}$$

For $t > \frac{1}{4}$ it holds

$$u(x,t) = \begin{cases} 4 & x \leq -\frac{1}{2}, \\ -4 & x > -\frac{1}{2}. \end{cases} \qquad \textbf{(3 points)}$$

Exercise 2: [3 points]

Given is the following differential equation for $u(x, y)$:

$$x \cdot u_{xx} - (x+y)u_{xy} + y \cdot u_{yy} = 0.$$

Provide the order of the differential equation and determine the type of the differential equation (elliptic, parabolic or hyperbolic) at the points $\binom{x_1}{y_1} = \binom{1}{1}$ and $\binom{x_2}{y_2} = \binom{1}{-1}$.

Solution:

The differential equation is of order two.

A type is obtained from the sign of

$D(x, y) = x \cdot y - \frac{(x+y)^2}{2^2} = -\frac{x^2 - 2xy + y^2}{4} = -\left(\frac{x-y}{2}\right)^2.$ **(1 point)**

$$\text{A differential equation is } \begin{cases} \text{elliptic} & \text{if } D(x,y) > 0, \\ \text{parabolic} & \text{if } D(x,y) = 0, \\ \text{hyperbolic} & \text{if } D(x,y) < 0. \end{cases}$$

$D(1, 1) = 1 \cdot 1 - \frac{(1+1)^2}{2^2} = 0 \Longrightarrow$ The differential equation is at $\binom{x_1}{y_1} = \binom{1}{1}$ parabolic.

$D(1, -1) = 1 \cdot (-1) - \frac{(1-1)^2}{2^2} = -1 \Longrightarrow$ The differential equation is at $\binom{x_2}{y_2} = \binom{1}{-1}$ hyperbolic.

(2 points)

Exercise 3: [4 points]

Let u be a harmonic function in the disc $\Omega := \{\binom{x}{y} \in \mathbb{R}^2 : x^2 + y^2 < 25\}$ with given values $g(x, y)$ on the boundary of the disc:

$$\begin{aligned} \Delta u(x,y) &= 0 \qquad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + y^2 < 25 \\ u(x,y) &= g(x,y) \qquad \text{for } \begin{pmatrix} x \\ y \end{pmatrix} \in \mathbb{R}^2 : x^2 + y^2 = 25. \end{aligned}$$

In the following two cases one can find solutions without long calculations. Give a solution for each case. Justify your answers.

a) $g(x, y) = \dfrac{x + y + 18}{9}$.

b) $g(x, y) = 2x^2 + 2y^2$.

Solution:

a) The function $u(x, y) = \dfrac{x + y + 18}{9} = g(x, y)$ solves the potential equation in the whole disc. Because of the uniqueness of the solution, $u(x, y) = \dfrac{x + y + 18}{9}$ is a unique solution in Ω .

(2 points)

b) $g(x, y) = 2x^2 + 2y^2 = 2(x^2 + y^2)$ is on the boundary $\partial\Omega$ constantly equal 50.

So $u(x, y)$ is constant on the boundary of Ω . Since the maximum and minimum of u in $\bar{\Omega}$ are attained on the boundary, u in the whole disc is constant and $u(x, y) = 50$.

(2 points)

Exercise 4: [6 points]

Determine the solution to the initial boundary value problem

$$\begin{aligned} u_{tt} - 36u_{xx} &= 0 && 0 < x < 2\pi\,,\ 0 < t\,, \\ u(x,0) &= 20\sin(\tfrac{3}{2}x) && 0 \le x \le 2\pi, \\ u_t(x,0) &= 24\sin(3x) && 0 \le x \le 2\pi, \\ u(0,t) &= 0 && 0 \le t, \\ u(2\pi,t) &= 0 && 0 \le t. \end{aligned}$$

Solution:

With $L = 2\pi$ and $c^2 = +\sqrt{36}$ the solution formula is:

$$u(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos\left(\frac{ck\pi}{L}t\right) + B_k \sin\left(\frac{ck\pi}{L}t\right)\right] \sin\left(\frac{k\pi}{L}x\right)$$

So

$$u(x,t) = \sum_{k=1}^{\infty} \left[A_k \cos(3kt) + B_k \sin(3kt)\right] \sin\left(\frac{k}{2}x\right). \qquad \textbf{(1 point)}$$

For $t = 0$ we have

$$u(x,0) = \sum_{k=1}^{\infty} A_k \sin\left(\frac{k}{2}x\right) \overset{!}{=} 20\sin(\frac{3}{2}x)$$

Hence $A_3 = 20$ and $A_k = 0$ else. **(2 points)**

$$u_t(x,t) = \sum_{k=1}^{\infty} \left[-A_k \cdot 3k \cdot \sin(3kt) + B_k \cdot 3k \cdot \cos(3kt)\right] \sin\left(\frac{k}{2}x\right)$$

and for $t = 0$:

$$u_t(x,t) = \sum_{k=1}^{\infty} 3kB_k \sin\left(\frac{k}{2}x\right) \overset{!}{=} 24\sin(3\pi x)$$

So $3 \cdot 6 \cdot B_6 \overset{!}{=} 24$ and $B_k = 0$ else. **(2 points)**

$$\begin{aligned} u(x,t) &= A_3 \cos(3\cdot 3t)\sin\left(\frac{3}{2}x\right) + B_6 \sin(3\cdot 6t)\sin\left(\frac{6}{2}x\right) \\ &= 20\cos(9t)\sin\left(\frac{3}{2}x\right) + \frac{4}{3}\sin(18t)\sin(3x) \qquad \textbf{(1 point)} \end{aligned}$$